GENERIC CONTROLLABILITY OF 3D SWIMMERS IN A PERFECT FLUID

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Abstract. We address the problem of controlling a dynamical system governing the motion of a 3D weighted shape changing body swimming in a perfect fluid. The rigid displacement of the swimmer results from the exchange of momentum between prescribed shape changes and the flow, the total impulse of the fluid-swimmer system being constant for all times. We prove the following tracking results: (i) Synchronized swimming: Maybe up to an arbitrarily small change of its density, any swimmer can approximately follow any given trajectory while, in addition, undergoing approximately any given shape changes. In this statement, the control consists in arbitrarily small superimposed deformations; (ii) Freestyle swimming: Maybe up to an arbitrarily small change of its density, any swimmer can approximately tracks any given trajectory by combining suitably at most five basic movements that can be generically chosen (no macro shape changes are prescribed in this statement).

Key words. Locomotion, Biomechanics, Ideal fluid, Geometric control theory

AMS subject classifications. 74F10, 70S05, 76B03, 93B27

1. Introduction.

1.1. Context. Researches on bio-inspired locomotion in fluid have now a long history. Focusing on the area of Mathematical Physics, the modeling leads to a system of PDEs (governing the fluid flow) coupled with a system of ODEs (driving the rigid motion of the immersed body). The first difficulty mathematicians came up against was to prove the well-posedness of such systems. This task was carried out in [16] (where the fluid is assumed to be viscous and governed by Navier Stokes equations), in [13] (for an inviscid fluid with potential flow) and in [5] (for low Reynolds numbers swimmers, the flow being governed by the stationary Stokes equations).

After the well-posedness of the fluid-swimmer dynamics were established, the following step was to investigate its controllability. On this topic, still few theoretical results are available: In [2], the authors prove that a 3D three-sphere mechanism, swimming along a straight line in a viscous fluid is controllable. In [4], we prove that a generic 2D example of shape changing body swimming in a potential flow can track approximately any given trajectory.

Some authors are rather interested in describing the dynamics of swimming in terms of Geometric Mechanics (within the general framework presented for instance in [12]). We refer to [7] and the very recent paper [8] for references in this area.

In this article, we consider a 3D shape changing body swimming in a potential flow. Under some symmetry assumptions (the swimmer is alone in the fluid and the fluid-swimmer system fills the whole space) we prove generic controllability results, generalizing and improving what has been obtained for a particular 2D model in [4].

1.2. Modeling.

Kinematics. We assume that the swimmer is the only immersed body in the fluid and that the fluid-swimmer system fills the whole space, identified with $\mathbf{R}^3$. Two frames are required in the modeling, the first one $\mathfrak{E} := (\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ is fixed and Galilean and the second one $\mathfrak{e} := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$

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is moving with its origin lying at any time at the center of mass of the swimming body. At any moment, there exist a rotation matrix $R \in \mathrm{SO}(3)$ and a vector $\mathbf{r} \in \mathbf{R}^3$ such that, if $X := (X_1, X_2, X_3)^t$ and $x := (x_1, x_2, x_3)^t$ are the coordinates of a same vector in respectively $\mathfrak{E}$ and $\mathfrak{e}$, then the equality $X = Rx + \mathbf{r}$ holds. The matrix R is supposed to give also the *orientation* of the swimmer. The rigid displacement of the swimmer, on a time interval $[0,T]$ ($T > 0$), is thoroughly described by the functions $t : [0,T] \mapsto R(t) \in \mathrm{SO}(3)$ and $t : [0,T] \mapsto \mathbf{r}(t) \in \mathbf{R}^3$, which are the unknowns of our problem. Denoting their time derivatives by $\dot{R}$ and $\dot{\mathbf{r}}$, we can define the linear velocity $\mathbf{v} := (v_1, v_2, v_3)^t \in \mathbf{R}^3$ and angular velocity vector $\boldsymbol{\Omega} := (\Omega_1, \Omega_2, \Omega_3)^t \in \mathbf{R}^3$ (both in $\mathfrak{e}$) by respectively $\mathbf{v} := R^t\dot{\mathbf{r}}$ and $\hat{\boldsymbol{\Omega}} := R^t\dot{R}$, where for every vector $\mathbf{u} := (u_1, u_2, u_3)^t \in \mathbf{R}^3$, $\hat{\mathbf{u}}$ is the unique skew-symmetric matrix satisfying $\hat{\mathbf{u}}x := \mathbf{u} \times x$ for every $x \in \mathbf{R}^3$. Vectors of $\mathbf{R}^6$ will sometimes be decomposed in the form $\mathbf{f} := (\mathbf{f}^1, \mathbf{f}^2)^t \in \mathbf{R}^3 \times \mathbf{R}^3$. For every $\mathbf{f} := (\mathbf{f}^1, \mathbf{f}^2) \in \mathbf{R}^6$ and $\mathbf{g} := (\mathbf{g}^1, \mathbf{g}^2) \in \mathbf{R}^6$, we can define $\mathbf{f} \star \mathbf{g} := (\mathbf{f}^1 \times \mathbf{f}^2, \mathbf{f}^1 \times \mathbf{g}^2 - \mathbf{g}^1 \times \mathbf{f}^2)^t \in \mathbf{R}^6$.

Shape Changes. Unless otherwise indicated, from now on all of the quantities will be expressed in the moving frame $\mathfrak{e}$. In our modeling, the domains occupied by the swimmer are images of the closed unit ball $\bar{B}$ by C^1 diffeomorphisms, isotopic the the identity, and tending to the identity at infinity, i.e. having the form $\mathrm{Id} + \vartheta$ where ϑ belongs to $D^1_0(\mathbf{R}^3)$ (definitions of the function spaces are given in the appendix, Section A). With these settings, the shape changes over a time interval $[0,T]$ can be simply prescribed by means of an absolutely continuous function $t \in [0,T] \mapsto \vartheta_t \in D^1_0(\mathbf{R}^3)$. Then, denoting $\Theta_t = \mathrm{Id} + \vartheta_t$, the domain occupied by the swimmer at the time $t \geq 0$ is the closed, bounded, connected set $\bar{\mathcal{B}}_t := \Theta_t(\bar{B})$ (do not forget that we are working in the frame $\mathfrak{e}$). We still require some notation: the unit ball's boundary is $\Sigma := \partial B$, $\Sigma_t := \Theta_t(\Sigma)$ stands for the body-fluid interface, $\mathbf{n}_t$ is the unit normal vector to Σ_t directed toward the interior of $\mathcal{B}_t$ and the fluid fills the exterior open set $\mathcal{F}_t := \mathbf{R}^3 \setminus \bar{\mathcal{B}}_t$.

So-called *self-propelled* constraints are necessary to ensure that the deformations result from the work of internal forces (they avoid for instance translations to be considered as allowable shape changes). Let a function $\varrho \in C^0(\bar{B})^+$ be given. The density of the deformed body at the instant t, denoted by $\varrho_t \in C^0(\bar{\mathcal{B}}_t)$, is defined by:

$$\varrho_t(x) := \varrho(\Theta_t^{-1}(x))/J_t(\Theta_t^{-1}(x)), \qquad (x \in \bar{\mathcal{B}}_t, \quad t \geq 0), \tag{1.1}$$

where $J_t := |\det \nabla\Theta_t| = \det \nabla\Theta_t$ (we can drop the absolute values here because $\Theta_t(x) \to x$ as $\|x\|_{\mathbf{R}^3} \to +\infty$ and $\det \nabla\Theta_t(x) \neq 0$ for all $x \in \mathbf{R}^3$ and $t \geq 0$). The self-propelled constraints read:

$$\int_{\mathcal{B}_t} \varrho_t(x)x \,\mathrm{d}x = \mathbf{0} \quad \text{and} \quad \int_{\mathcal{B}_t} \varrho_t(x)\partial_t\Theta_t(\Theta_t^{-1}(x)) \times x \,\mathrm{d}x = \mathbf{0} \quad (t \geq 0). \tag{1.2a}$$

The former identity means that, as already mentioned before, the center of mass of the swimmer lies at any time at the origin of the moving frame. The latter relation tells us that the angular momentum (in $\mathfrak{e}$) has to remain constant as the swimmer undergoes shape changes. Equivalent formulations can be obtained up to a change of variables:

$$\int_B \varrho(x)\Theta_t(x) \,\mathrm{d}x = \mathbf{0} \quad \text{and} \quad \int_B \varrho(x)\partial_t\Theta_t(x) \times \Theta_t(x) \,\mathrm{d}x = \mathbf{0} \quad (t \geq 0). \tag{1.2b}$$

The Flow. The fluid is assumed to be inviscid and incompressible. We denote by $\varrho_f > 0$ its constant density. The flow is governed by Euler equations. According to Helmholtz's third

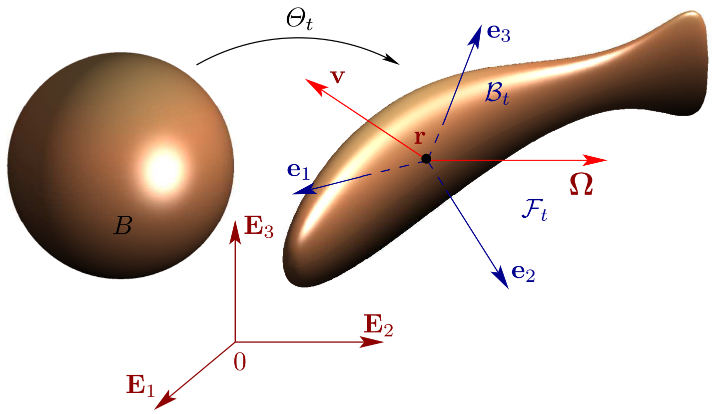

FIG. 1.1. *Kinematics of the model: The Galilean frame* $\mathfrak{E} := (\mathbf{E}_j)_{1\leq j\leq 3}$ *and the moving frame* $\mathfrak{e} := (\mathbf{e}_j)_{1\leq j\leq 3}$ *with* $\mathbf{e}_j = R\,\mathbf{E}_j$ $(R \in \mathrm{SO}(3))$*. Quantities are usually expressed in the moving frame. The domain of the body is* $\bar{\mathcal{B}}_t$ *at the time* t *and* $\mathcal{B}_t$ *is the image of the unit ball* B *by a diffeomorphism* Θ_t*. The open set* $\mathcal{F}_t := \mathbf{R}^3 \setminus \bar{\mathcal{B}}_t$ *is the domain of the fluid. The center of mass of the body is denoted* $\mathbf{r}$ *(in* $\mathfrak{E}$*),* $\mathbf{v} := R^t\dot{\mathbf{r}}$ *is its translational velocity (in* $\mathfrak{e}$*) and* $\mathbf{\Omega}$ *its angular velocity.*

theorem, if the flow is irrotational at the initial time, it remains irrotational for all times. In this case, the Eulerian velocity is equal at any time to the gradient of a potential function. According to Kirchhoff's law, the potential can be decomposed into a linear combination of elementary potentials, each one connected to a degree of freedom of the system (they consist here in the 6 degrees of freedom of the rigid motion of the body plus those connecting to the deformations). These ideas have been thoroughly described in a series of papers [7, 13, 14, 4], to which we refer for further explanations.

The elementary potentials ψ^i_t, $(i = 1,\ldots,6)$ corresponding to the rigid motion of the swimmer are harmonic in $\mathcal{F}_t$, tend to zero as infinity and satisfy the Neumann boundary conditions $\partial_n\psi^i_t = (\mathbf{e}_i \times x)\cdot\mathbf{n}_t$ $(i = 1,2,3)$ and $\partial_n\psi^i_t = \mathbf{e}_{i-3}\cdot\mathbf{n}_t$ $(i = 4,5,6)$ on Σ_t. They are well defined in the weighted Sobolev space $W^1(\mathcal{F}_t)$ (defined in the appendix, Section A; see also [3] for details). The overall potential connecting to the rigid displacement is $\psi_t := \sum_{i=1}^3 \Omega_i\psi^i_t + \sum_{i=4}^5 v_{i-3}\psi^i_t$ $(t \geq 0)$. On the other hand, the elementary potential φ_t associated to the shape changes, harmonic as well in $\mathcal{F}_t$, satisfies the boundary condition $\partial_n\varphi_t = \mathbf{w}_t\cdot\mathbf{n}_t$ on Σ_t $(i = 1,\ldots,n)$, where $\mathbf{w}_t(x) := \partial_t\Theta_t(\Theta_t^{-1}(x))$ $(x \in \mathbf{R}^3)$. Like the functions ψ^i_t $(i = 1,2,3)$, φ_t belongs to $W^1(\mathcal{F}_t)$ for all $t > 0$.

Dynamics. The modeling of moving rigid (or shape changing) bodies in an ideal fluid classically involves the notion of mass matrices. The mass of the body is $m := \int_B \varrho\,\mathrm{d}x$ and its inertia tensor at the time $t \geq 0$ is defined by:

$$\mathbb{I}(t) := \int_{\mathcal{B}_t} \varrho_t\big[\|x\|^2_{\mathbf{R}^3}\mathrm{Id} - x\otimes x\big]\,\mathrm{d}x = \int_B \varrho\big[\|\Theta_t(x)\|^2_{\mathbf{R}^3}\mathrm{Id} - \Theta_t(x)\otimes\Theta_t(x)\big]\,\mathrm{d}x. \tag{1.3}$$

We introduce $\mathbb{M}^r_b(t) := \mathrm{diag}(\mathbb{I}(t), m\mathrm{Id})$ (a 6×6 symmetric bloc diagonal matrix), $\mathbb{M}^r_f(t)$ (a 6×6 symmetric matrix as well) whose entries read:

$$\varrho_f \int_{\mathcal{F}_t} \nabla\psi^i_t\cdot\nabla\psi^j_t\,\mathrm{d}x, \quad (1 \leq i,j \leq 6), \tag{1.4}$$

and we denote $\mathbb{M}^r(t) := \mathbb{M}^r_b(t) + \mathbb{M}^r_f(t)$. We also need the 6×1 column vector $\mathbf{N}(t)$, homogeneous to a momentum, whose elements read:

$$\varrho_f \int_{\mathcal{F}_t} \nabla \psi^i_t \cdot \nabla \varphi_t \, \mathrm{d}x, \quad (1 \leq i \leq 6). \tag{1.5}$$

If we neglect the buoyancy force, it has been proved in a series of papers (we refer for instance to the already mentioned articles [7, 14] or [4]) that the swimming motion is governed by the equation:

$$\begin{pmatrix} \mathbf{\Omega} \\ \mathbf{v} \end{pmatrix} = -\mathbb{M}^r(t)^{-1}\mathbf{N}(t), \qquad (t \geq 0). \tag{1.6a}$$

At this point, we can identified ϱ_t (or more simply ϱ since they are linked through relation (1.1)) as a parameter and the control as being the function $t \in [0, T] \mapsto \vartheta_t \in D^1_0(\mathbf{R}^3)$. Notice that the dependance of the dynamics in the control is strongly nonlinear. Indeed ϑ_t describes the shape of the body and hence also the domain of the fluid in which are set the PDEs of the potential functions involved in the expressions of the mass matrices $\mathbb{M}^r(t)$ and $\mathbf{N}(t)$.

To determine the rigid motion, Equation (1.6a) has to be supplemented with the ODE:

$$\frac{d}{dt}\begin{pmatrix} R \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} R\,\hat{\mathbf{\Omega}} \\ R\,\mathbf{v} \end{pmatrix}, \qquad (t > 0), \tag{1.6b}$$

together with Cauchy data for $R(0)$ and $\mathbf{r}(0)$. Remark that the dynamics does not depend on ϱ and ϱ_f independently but only on the relative density ϱ/ϱ_f. So we can assume, without loss of generality, that $\varrho_f = 1$ in the sequel.

1.3. Main results. The first result ensures the well posedness of System (1.6) and the continuity of the input-output mapping:

Proposition 1.1. *For any $T > 0$, any $\varrho \in C^0(\bar{B})^+$, any absolutely continuous function $t \in [0, T] \mapsto \vartheta_t \in D^1_0(\mathbf{R}^3)$ (respectively of class C^p, $p = 1, \ldots, +\infty$ or analytic) and any inital data $(R(0), \mathbf{r}(0)) \in \mathrm{SO}(3) \times \mathbf{R}^3$, System (1.6) admits a unique solution $t \in [0, T] \mapsto (R(t), \mathbf{r}(t)) \in \mathrm{SO}(3) \times \mathbf{R}^3$ (in the sense of Carathéodory) absolutely continuous on $[0, T]$ (respectively of class C^p or analytic).*

Let $t \in [0, T] \mapsto \vartheta^j_t \in D^1_0(\mathbf{R}^3)$ (for $j = 1, \ldots, +\infty$) be a sequence of controls in $AC([0, T], D^1_0(\mathbf{R}^3))$ (see Section A for a definition of this space) which converges in this space to a function $t \in [0, T] \mapsto \bar{\vartheta}_t \in D^1_0(\mathbf{R}^3)$. Let a pair $(R_0, \mathbf{r}_0) \in \mathrm{SO}(3) \times \mathbf{R}^3$ be given and denote $t \in [0, T] \mapsto (\bar{R}(t), \bar{\mathbf{r}}(t)) \in \mathrm{SO}(3) \times \mathbf{R}^3$ the solution in $AC([0, T], \mathrm{SO}(3) \times \mathbf{R}^3)$ to System (1.6) with control $\bar{\vartheta}$ and Cauchy data $(R_0, \mathbf{r}_0)$. Then, the unique solution $(R^j, \mathbf{r}^j)$ to System (1.6) with control ϑ^j and Cauchy data $(R_0, \mathbf{r}_0)$ converges in $AC([0, T], \mathrm{SO}(3) \times \mathbf{R}^3)$ to $(\bar{R}, \bar{\mathbf{r}})$ as $j \to +\infty$.

We denote by $\mathrm{M}(3)$ the Banach space of the 3×3 matrices endowed wit any matrix norm. The main result of this article addresses the controllability of System (1.6):

Theorem 1.2. *(Synchronized Swimming) Assume that the following data are given: (i) A function $\bar{\varrho}$ in $C^0(\bar{B})^+$ (the target density of the swimmer); (ii) A C^1 function $t \in [0, T] \mapsto \bar{\vartheta}_t \in D^1_0(\mathbf{R}^3)$ (the target shape changes) such that the pair $(\bar{\varrho}, \bar{\vartheta})$ satisfies the self-propelled constraints (1.2); (iii) A C^1 function $t \in [0, T] \mapsto (\bar{R}(t), \bar{\mathbf{r}}(t)) \in \mathrm{SO}(3) \times \mathbf{R}^3$ (the target trajectory to be followed). Then, for any $\varepsilon > 0$, there exists a function $\varrho \in C^0(\bar{B})^+$ (the actual density of the swimmer) and a function $t \in [0, T] \mapsto \vartheta_t \in D^1_0(\mathbf{R}^3)$ (the actual shape changes that can be chosen of class C^p for any*

$p = 1, \dots, +\infty$ *or even analytic) such that the pair* (ϱ, ϑ) *satisfies* (1.2), $\|\bar\varrho - \varrho\|_{C^0(\bar B)} < \varepsilon$, $\vartheta_0 = \bar\vartheta_0$, $\vartheta_T = \bar\vartheta_T$ *and* $\sup_{t\in[0,T]}\Big(\|\bar\vartheta_t - \vartheta_t\|_{C^1_0(\mathbf{R}^3)^3} + \|\bar R(t) - R(t)\|_{\mathrm{M}(3)} + \|\bar{\mathbf{r}}(t) - \mathbf{r}(t)\|_{\mathbf{R}^3}\Big) < \varepsilon$ *where the function* $t \in [0,T] \mapsto (R(t), \mathbf{r}(t)) \in \mathrm{SO}(3)\times\mathbf{R}^3$ *is the unique solution to system* (1.6) *with initial data* $(R(0), \mathbf{r}(0)) = (\bar R(0), \bar{\mathbf{r}}(0))$ *and control* ϑ*.*

This theorem tells us that, maybe up to an arbitrarily small change of its density, any weighted 3D body undergoing approximately any prescribed shape changes can approximately track by swimming any given trajectory. It may seem surprising that the shape changes, which are supposed to be the control of our problem, can also be prescribed. Actually, be aware that they are only *approximately* prescribed. We are going to show precisely that arbitrarily small superimposed shape changes suffice for controlling the swimming motion. This result improves what has been done in the article [4] for a particular 2D model.

When no macro shape changes are prescribed we have:

THEOREM 1.3. *(Freestyle Swimming) Assume that the following data are given: (i) A pair* $(\bar\varrho, \bar\vartheta) \in C^0(\bar B)^+ \times D^1_0(\mathbf{R}^3)$ *satisfying* (1.2) *(the target density and shape at rest) (ii) A* C^1 *function* $t \in [0,T] \mapsto (\bar R(t), \bar{\mathbf{r}}(t)) \in \mathrm{SO}(3)\times\mathbf{R}^3$ *(the target trajectory). Then, for any* $\varepsilon > 0$ *there exists a pair* $(\varrho, \vartheta) \in C^0(\bar B)^+ \times D^1_0(\mathbf{R}^3)$ *(the actual density and shape at rest) such that (i)* $\int_B \varrho\, \Theta\, \mathrm{d}x = \mathbf{0}$ *(where* $\Theta := \mathrm{Id} + \vartheta$*) (ii)* $\|\bar\varrho - \varrho\|_{C^0(\bar B)} + \|\bar\vartheta - \vartheta\|_{D^1_0(\mathbf{R}^3)} < \varepsilon$ *and (iii) for almost any* 5*-uplet* $(\mathbf{V}_1, \dots, \mathbf{V}_5) \in (C^1_0(\mathbf{R}^3)^3)^5$ *satisfying* $\int_B \varrho\, \mathbf{V}_i\, \mathrm{d}x = \mathbf{0}$, $\int_B \varrho\, \Theta \times \mathbf{V}_i\, \mathrm{d}x = \mathbf{0}$ *and* $\int_B \varrho\, \mathbf{V}_i \times \mathbf{V}_j\, \mathrm{d}x = \mathbf{0}$ $(i, j = 1, \dots, 5)$*, there exists a function* $t \in [0,T] \mapsto s(t) \in \mathbf{R}^5$ *(that can be chosen of class* C^p *for any* $p = 1, \dots, +\infty$ *or even analytic) such that, using* $\vartheta_t := \vartheta + \sum_{i=1}^5 s_i(t)\mathbf{V}_i \in D^1_0(\mathbf{R}^3)$ *as control in the dynamics* (1.6)*, we get* $\sup_{t\in[0,T]}\big(\|\bar R(t) - R(t)\|_{\mathrm{M}(3)} + \|\bar{\mathbf{r}}(t) - \mathbf{r}(t)\|_{\mathbf{R}^3}\big) < \varepsilon$ *where the function* $t \in [0,T] \mapsto (R(t), \mathbf{r}(t)) \in \mathrm{SO}(3)\times\mathbf{R}^3$ *is the unique solution to ODEs* (1.6) *with initial data* $(R(0), \mathbf{r}(0)) = (\bar R(0), \bar{\mathbf{r}}(0))$*.*

Differently stated, we claim in this Theorem that any weighted 3D body (maybe up to an arbitrarily small change of its density) is able to swim by means of allowable deformations (i.e. satisfying the self-propelled constraints) obtained as a suitable combination of pretty much any given five basic movements.

The proofs rely on the following main ideas: First, we shall identify a set of parameters necessary to thoroughly characterize a swimmer and its way of swimming (these parameters are its density, its shape and a finite number of basic movements, satisfying the self-propelled constraints (1.2)). Any set of such parameters will be termed a *swimmer configuration* (denoted SC in short). Then, the set of all of the SC will be shown to be an (infinite dimensional) analytic connected embedded submanifold of a Banach space.

The second step of the reasoning will consist in proving that the swimmer's ability to track any given trajectory (while undergoing any given shape changes) is related to the vanishing of some analytic functions depending on the SC. These functions are connected to the determinant of some vector fields and their Lie brackets (we will use here some classical result of Geometric Control Theory). Eventually, by direct calculation, we will prove that at least one swimmer (corresponding to one particular SC) has this ability. An elementary property of analytic functions will eventually allow us to conclude that almost any SC (or equivalently any swimmer) has this property.

REMARK 1.4. *The authors conjecture that in both Theorem 1.2 and Theorem 1.3, the actual density* ϱ *can be chosen equal to the target density* $\bar\varrho$*. At this point however and although it is very unlikely, it can not be excluded that all of the swimmers with a particular density might be unable to swim. This issue also appeared in [4].*

1.4. Outline of the paper. The next Section is dedicated to the notion of swimmer configuration (definition and properties). In Section 3 we show that the mass matrices are analytic functions in the SC (seen as a variable) and in Section 4 we will restate the control problem in order to fit with the general framework of Geometric Control Theory. A particular case of swimmer will be shown to be controllable. In Section 5 the proof of the main results will be performed. Section 6 contains some words of conclusion. Many technical results are gathered in the appendix to avoid overloading the rest of the paper.

2. Swimmer Configuration. A *swimmer configuration* is a set of parameters characterizing swimmers whose deformations consist in a combination of a finite number of basic movements.

Definition 2.1. *For any positive integer n, we denote $\mathcal{C}(n)$ the subset of $C^0(\bar{B})^+ \times D_0^1(\mathbf{R}^3) \times (C_0^1(\mathbf{R}^3)^3)^n$ consisting of all of the triplets $c := (\varrho, \vartheta, \mathcal{V})$ such that, denoting $\Theta := \mathrm{Id} + \vartheta$ and $\mathcal{V} := (\mathbf{V}_1, \dots, \mathbf{V}_n)$, the following conditions hold (i) the set $\{\mathbf{V}_i|_{\bar{B}} \cdot \mathbf{e}_k,\ 1 \leq i \leq n,\ k = 1,2,3\}$ is a free family in $C_0^1(\bar{B})$ (ii) every pair $(\mathbf{V}, \mathbf{V}')$ of elements of $\{\Theta, \mathbf{V}_1, \dots, \mathbf{V}_n\}$ satisfies $\int_B \varrho \mathbf{V}\, \mathrm{d}x = \mathbf{0}$ and $\int_B \varrho \mathbf{V} \times \mathbf{V}'\, \mathrm{d}x = \mathbf{0}$.*

We call swimmer configuration (SC in short) any element c of $\mathcal{C}(n)$.

By definition, $D_0^1(\mathbf{R}^3)$ is open in $C_0^1(\mathbf{R}^3)^3$ (see appendix, Section A). We deduce that for any $c \in \mathcal{C}(n)$, the set $\{s := (s_1, \dots, s_n)^t \in \mathbf{R}^n \,:\, \vartheta + \sum_{i=1}^n s_i \mathbf{V}_i \in D_0^1(\mathbf{R}^3)\}$ is open as well in $\mathbf{R}^n$ and we denote $\mathcal{S}(c)$ its connected component containing $s = 0$.

Definition 2.2. *For any positive integer n, we call extended swimmer configuration (ESC in short) any pair $\mathbf{c} := (c, s)$ such that $c \in \mathcal{C}(n)$ and $s \in \mathcal{S}(c)$. We denote $\mathcal{C}_X(n)$ the set of all of these pairs.*

Restatement of the problem in terms of SC and ESC. Pick a SC in $\mathcal{C}(n)$ (for some integer n). The characteristics of the corresponding swimmer can be deduced from c as follows: If c is equal to $(\varrho, \vartheta, \mathcal{V})$ with $\mathcal{V} := (\mathbf{V}_1, \dots, \mathbf{V}_n)$, the shape of the swimmer at rest is $\bar{\mathcal{B}} := \Theta(\bar{B})$ where $\Theta := \mathrm{Id} + \vartheta$. When swimming, it can occupy the domains $\bar{\mathcal{B}}_{\mathbf{c}} := \Theta_s(\bar{B})$ for all $s \in \mathcal{S}(c)$ ($\mathbf{c} := (c, s) \in \mathcal{C}(n)$ is hence an ESC), where $\Theta_s := \mathrm{Id} + \vartheta + \sum_{i=1}^n s_i \mathbf{V}_i$. Still for any $s \in \mathcal{S}(c)$, its density is the function $\varrho_s \in C^0(\bar{\mathcal{B}}_{\mathbf{c}})^+$ defined by $\varrho_s(x) := \varrho(\Theta_s^{-1}(x))/J_s(\Theta_s^{-1}(x))$ with $J_s := \det \nabla \Theta_s$. Notice that within this construction, the shape changes on a time interval $[0, T]$ ($T > 0$) are merely given through an absolutely continuous function $t : [0,T] \mapsto s(t) \in \mathcal{S}(c)$. If $t \in [0,T] \mapsto \dot{s}(t) \in \mathbf{R}^n$ stands for its time derivative, the Lagrangian velocity at a point x of $\bar{B}$ is $\sum_{i=1}^n \dot{s}_i(t) \mathbf{V}_i(x)$ while the Eulerian velocity at a point $x \in \bar{\mathcal{B}}_{\mathbf{c}}$ is $\sum_{i=1}^n \dot{s}_i(t) \mathbf{w}_s^i(x)$ where $\mathbf{w}_s^i(x) := \mathbf{V}_i(\Theta_s^{-1}(x))$. Due to assumption (ii) of Definition 2.1, the self-propelled constraints (1.2) are automatically satisfied.

The harmonic elementary potential functions of the fluid corresponding to the rigid motions depend only on the ESC. Therefore, they will be denoted in the sequel $\psi_{\mathbf{c}}^i$ instead of ψ_t^i. The same remark holds for the inertia tensor $\mathbb{I}(t)$ and the mass matrices $\mathbb{M}^r(t)$, $\mathbb{M}_b^r(t)$ and $\mathbb{M}_f^r(t)$ whose notation is turned into $\mathbb{I}(\mathbf{c})$, $\mathbb{M}^r(\mathbf{c})$, $\mathbb{M}_b^r(\mathbf{c})$ and $\mathbb{M}_f^r(\mathbf{c})$ respectively. The elementary potential connected to the shape changes can be decomposed into $\sum_{i=1}^n \dot{s}_i \varphi_{\mathbf{c}}^i$. In this sum, each potential function $\varphi_{\mathbf{c}}^i$ is harmonic in $\mathcal{F}_{\mathbf{c}} := \mathbf{R}^3 \setminus \bar{\mathcal{B}}_{\mathbf{c}}$ and satisfies on $\Sigma_{\mathbf{c}} := \partial \mathcal{B}_{\mathbf{c}}$ the Neumann boundary conditions $\partial_n \varphi_{\mathbf{c}}^i = \mathbf{w}_s^i \cdot \mathbf{n}_{\mathbf{c}}$, $\mathbf{n}_{\mathbf{c}}$ being the unit normal to $\Sigma_{\mathbf{c}}$ directed toward the interior of $\mathcal{B}_{\mathbf{c}}$. Introducing the mass matrix $\mathbb{N}(\mathbf{c})$, whose elements are $\varrho_f \int_{\mathcal{B}_{\mathbf{c}}} \nabla \psi_{\mathbf{c}}^i \cdot \nabla \varphi_{\mathbf{c}}^j \mathrm{d}x$ ($1 \leq i \leq 6$, $1 \leq j \leq n$) (recall that ϱ_f can be chosen equal to 1), the dynamics (1.6a) can now be rewritten in the form:

$$\begin{pmatrix} \mathbf{\Omega} \\ \mathbf{v} \end{pmatrix} = -\mathbb{M}^r(\mathbf{c})^{-1} \mathbb{N}(\mathbf{c}) \dot{s}, \qquad (t \geq 0). \tag{2.1}$$

Let us focus on the properties of $\mathcal{C}(n)$ and $\mathcal{C}_X(n)$.

THEOREM 2.3. *For any positive integer n, the set $\mathcal{C}(n)$ is an analytic connected embedded submanifold of $C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times (C^1_0(\mathbf{R}^3)^3)^n$ of codimension $N := 3(n+2)(n+1)/2$.*

The definition and the main properties of analytic functions in Banach spaces are summarized in the article [17].

Proof. For any $c := (\varrho,\vartheta,\mathcal{V})\in C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times (C^1_0(\mathbf{R}^3)^3)^n$, denote $\mathbf{V}_0 := \mathrm{Id}+\vartheta$ and $\mathcal{V} := (\mathbf{V}_1,\ldots,\mathbf{V}_n)$. Then, define for $k=0,1,\ldots,n$, the functions $\Lambda_k : C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times (C^1_0(\mathbf{R}^3)^3)^n\to \mathbf{R}^{3(n+1-k)}$ by $\Lambda_k(c) := \Big(\int_B \varrho\,\mathbf{V}_k\,\mathrm{d}x,\ \int_B\varrho\,\mathbf{V}_k\times\mathbf{V}_{k+1}\,\mathrm{d}x,\ \ldots,\int_B\varrho\,\mathbf{V}_k\times\mathbf{V}_n\,\mathrm{d}x\Big)^t$. Every function Λ_k is analytic and we draw the same conclusion for $\Lambda := (\Lambda_0,\ldots,\Lambda_n)^t : C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times(C^1_0(\mathbf{R}^3)^3)^n\to\mathbf{R}^N$ ($N := 3(n+2)(n+1)/2$). In order to prove that $\partial_c\Lambda(c)$ (the differential of Λ at the point c) is onto for any $c\in\mathcal{C}(n)$, assume that there exist $(n+2)(n+1)/2$ vectors $\boldsymbol{\alpha}_i^j\in\mathbf{R}^3$ ($0\leq i\leq j\leq n$) such that:

$$\sum_{i=0}^n \boldsymbol{\alpha}_i\cdot\langle\partial_c\Lambda(c),c^h\rangle = \mathbf{0},\qquad \forall\, c^h\in C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times(C^1_0(\mathbf{R}^3))^3, \tag{2.2}$$

where $\boldsymbol{\alpha}_i := (\boldsymbol{\alpha}_i^i,\boldsymbol{\alpha}_i^{i+1},\ldots,\boldsymbol{\alpha}_i^n)^t\in\mathbf{R}^{3(n+1-i)}$ ($j=0,\ldots,n$) and $c^h := (\varrho^h,\vartheta^h,\mathcal{V}^h)\in C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times(C^1_0(\mathbf{R}^3)^3)^n$ with $\mathbf{V}_0^h := \mathrm{Id}+\vartheta^h$ and $\mathcal{V}^h := (\mathbf{V}_1^h,\ldots,\mathbf{V}_n^h)$. Reorganizing the terms in (2.2), we obtain that:

$$\int_B \varrho^h\Big[\sum_{k=0}^n\boldsymbol{\alpha}_k^k\cdot\mathbf{V}_k+\sum_{0\leq i<j\leq n}\boldsymbol{\alpha}_i^j\cdot(\mathbf{V}_i\times\mathbf{V}_j)\Big]\mathrm{d}x+$$
$$\sum_{k=0}^n\int_B\varrho\mathbf{V}_k^h\cdot\Big[\sum_{j=0}^{k-1}\boldsymbol{\alpha}_j^k\times\mathbf{V}_j+\boldsymbol{\alpha}_k^k-\sum_{j=k+1}^n\boldsymbol{\alpha}_k^j\times\mathbf{V}_j\Big]\mathrm{d}x=0.$$

Since this identity has to be satisfied for any $(\varrho^h,\vartheta^h,\mathcal{V}^h)\in C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times(C^1_0(\mathbf{R}^3))^3$, we deduce that, for every $k=0,\ldots,n$:

$$\sum_{p=0}^n\boldsymbol{\alpha}_p^p\cdot\mathbf{V}_p+\sum_{0\leq i<j\leq n}\boldsymbol{\alpha}_i^j\cdot(\mathbf{V}_i\times\mathbf{V}_j)=0\quad\text{and}\quad\sum_{j=0}^{k-1}\boldsymbol{\alpha}_j^k\times\mathbf{V}_j+\boldsymbol{\alpha}_k^k-\sum_{j=k+1}^n\boldsymbol{\alpha}_k^j\times\mathbf{V}_j=\mathbf{0}. \tag{2.3}$$

Multiplying the latter equality by ϱ and integrating over B, we get that $\boldsymbol{\alpha}_k^k=\mathbf{0}$ ($k=0,\ldots,n$). The first equality in (2.3) is now just a linear combination of the second ones (for $k=0,\ldots,n$), so we can drop it. Taking into account Hypothesis (ii) of Definition 2.1, latter identity in (2.3) with $k=0$ leads to $\boldsymbol{\alpha}_0^j=\mathbf{0}$ for every $j=1,\ldots,n$. There are no more terms involving $\mathbf{V}_0$ in the other equations and invoking again Hypothesis (ii) we eventually get $\boldsymbol{\alpha}_i^j=\mathbf{0}$ for $1\leq i<j\leq n$. So, equality (2.2) entails that $\boldsymbol{\alpha}_i=\mathbf{0}$ for all $i=0,\ldots,n$ and the mapping $\partial_c\Lambda(c)$ is onto for all $c\in\mathcal{C}(n)$.

The linear space $X=\mathrm{Ker}\,\partial_c\Lambda(c)$ is closed since Λ is analytic. Let Y be an algebraic supplement of X in $C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times(C^1_0(\mathbf{R}^3)^3)^n$, and denote by Π_Y the linear projection onto Y along X. A crucial observation is that the linear space Y is isomorphic to $\mathbf{R}^N$ and hence it is finite dimensional and closed in $C^0(\bar{B})\times C^1_0(\mathbf{R}^3)^3\times(C^1_0(\mathbf{R}^3)^3)^n$. Define the analytic mapping $f: X\times Y\to\mathbf{R}^N$ by $f(x,y)=\Lambda(c+x+y)$. The mapping $\partial_y f(0,0)=\partial_c\Lambda(c)\circ\Pi_Y$ being onto, the implicit function theorem (analytic version in Banach spaces, see [17]) asserts that there exist an open neighborhood

$\mathcal{O}_1$ of 0 in X, an open neighborhood $\mathcal{O}_2$ of 0 in Y, and an analytic mapping $g : \mathcal{O}_1 \to Y$ such that $g(0) = 0$ and, for every (x, y) in $\mathcal{O}_1 \times \mathcal{O}_2$, the two following assertions are equivalent: (i) $f(x, y) = 0$ (or, in other words, $c + x + y$ belongs to $\mathcal{C}(n)$), and (ii) $y = g(x)$. The analytic mapping g provides a local parameterization of $\mathcal{C}(n)$ in a neighborhood of c.

In order to prove that $\mathcal{C}(n)$ is path-connected, consider two elements $c^\dagger := (\varrho^\dagger, \vartheta^\dagger, \mathcal{V}^\dagger)$ and $c^\ddagger := (\varrho^\ddagger, \vartheta^\ddagger, \mathcal{V}^\ddagger)$ of $\mathcal{C}(n)$ and denote $\Theta^\dagger := \mathrm{Id} + \vartheta^\dagger$, $\mathcal{V}^\dagger := (\mathbf{V}_1^\dagger, \ldots, \mathbf{V}_n^\dagger)$ and $\Theta^\ddagger := \mathrm{Id} + \vartheta^\ddagger$, $\mathcal{V}^\ddagger := (\mathbf{V}_1^\ddagger, \ldots, \mathbf{V}_n^\ddagger)$. According to Definition A.2, $D_0^1(\mathbf{R}^3)$ is open and connected. It entails that it is always possible to find a continuous, piecewise linear path $t : [0, 1] \mapsto \bar\vartheta_t \in D_0^1(\mathbf{R}^3)$ such that $\bar\vartheta_{t=0} = \vartheta^\dagger$ and $\bar\vartheta_{t=1} = \vartheta^\ddagger$. We introduce $0 = t_0 < t_1 < \ldots < t_k = 1$, a subdivision of the interval $[0, 1]$ such that $t \mapsto \bar\vartheta_t$ is linear on every subinterval $[t_j, t_{j+1}]$ ($j = 0, \ldots, k-1$) and we denote $\bar\Theta_t := \mathrm{Id} + \bar\vartheta_t$, $\bar\vartheta^j := \bar\vartheta_{t=t_j}$, $\bar\Theta^j := \mathrm{Id} + \bar\vartheta^j$ ($j = 0, \ldots, k$). Let us introduce as well the continuous functions $t \in [0, 1] \mapsto \varrho_t := t\varrho^\ddagger + (1-t)\varrho^\dagger \in C^0(\bar B)$ and $t \in [0, 1] \mapsto \mathbf{u}(t) := -\int_B \varrho_t \bar\Theta_t \mathrm{d}x / \int_B \varrho_t \mathrm{d}x \in \mathbf{R}^3$. The set $\cup_{t\in[0,T]} \bar\Theta_t(\bar B) + \mathbf{u}(t)$ being compact, it is contained in a large ball Ω. We introduce Ω' an even larger ball containing Ω and a cut-off function χ defined in $\mathbf{R}^3$ such that $0 \leq \chi \leq 1$, $\chi = 1$ in Ω and $\chi = 0$ in $\mathbf{R}^3 \setminus \bar\Omega'$. The derivative $\dot{\mathbf{u}}$ of $\mathbf{u}$ exists everywhere on $]0, T[$ excepted maybe at the points $t_1, \ldots, t_{k-1}$. The flow associated with the Carathéodory's solutions of the Cauchy problem: $\dot X_t(x) = \dot{\mathbf{u}}(t)\chi(X_t(x))$, $(t > 0)$, $X_{t=0}(x) = x$ is well defined (see [11, Theorem 1A, page 57]). Moreover, for every fixed $t \in [0, 1]$, the mapping $x \in \mathbf{R}^3 \mapsto X_t(x) \in \mathbf{R}^3$ is C^∞. Consider now the mappings $t \in [0, 1] \mapsto \vartheta_t := X_t \circ \bar\Theta_t - \mathrm{Id}$ and $\Theta_t := \mathrm{Id} + \vartheta_t$. If $x \in \mathbf{R}^3 \setminus \bar\Omega'$, $\Theta_t(x) = \bar\Theta_t(x)$ for all $t \in [0, T]$ and if $x \in \bar B$ then $\vartheta_t(x) = \bar\vartheta_t(x) + \mathbf{u}(t)$ and $\Theta_t(x) = \bar\Theta_t(x) + \mathbf{u}(t)$. Notice that $\vartheta_t \in D_0^1(\mathbf{R}^3)$ for all $t \in [0, T]$ and $\int_B \varrho_t(x)\Theta_t(x)\mathrm{d}x = 0$ for all $t \in [0, T]$. Since $C_0^1(\mathbf{R}^3)^3$ is an infinite dimensional Banach space, it is always possible to find by induction $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n$ in $C_0^1(\mathbf{R}^3)^3$ such that (i) both families $\{\mathbf{W}_1|_{\bar B} \cdot \mathbf{e}_k, \ldots, \mathbf{W}_n|_{\bar B} \cdot \mathbf{e}_k, \mathbf{V}_1^\dagger|_{\bar B} \cdot \mathbf{e}_k, \ldots, \mathbf{V}_n^\dagger|_{\bar B} \cdot \mathbf{e}_k,\ k = 1, 2, 3\}$ and $\{\mathbf{W}_1|_{\bar B} \cdot \mathbf{e}_k, \ldots, \mathbf{W}_n|_{\bar B} \cdot \mathbf{e}_k, \mathbf{V}_1^\ddagger|_{\bar B} \cdot \mathbf{e}_k, \ldots, \mathbf{V}_n^\ddagger|_{\bar B} \cdot \mathbf{e}_k,\ k = 1, 2, 3\}$ are free in $C_0^1(\mathbf{R}^3)$ and (ii) for any pair of elements $\mathbf{V}$, $\mathbf{V}'$(both picked in the same family, $\int_B \varrho^\dagger \mathbf{V}\mathrm{d}x = \mathbf{0}$, $\int_B \varrho^\ddagger \mathbf{V}\mathrm{d}x = \mathbf{0}$, $\int_B \varrho^\dagger \bar\Theta^j \times \mathbf{V}\mathrm{d}x = \mathbf{0}$, $\int_B \varrho^\ddagger \bar\Theta^j \times \mathbf{V}\mathrm{d}x = \mathbf{0}$ (for all $j = 1, \ldots, k$), $\int_B \varrho^\dagger \mathbf{V} \times \mathbf{V}'\mathrm{d}x = \mathbf{0}$ and $\int_B \varrho^\ddagger \mathbf{V} \times \mathbf{V}'\mathrm{d}x = \mathbf{0}$. Define the function $t \in [0, 1] \mapsto \mathbf{V}_t^i \in C_0^1(\mathbf{R}^3)^3$ by $\mathbf{V}_t^i := (1 - 2t)\mathbf{V}_i^\dagger + 2t\mathbf{W}_i$ if $0 \leq t \leq 1/2$ and $\mathbf{V}_t^i := (2 - 2t)\mathbf{W}_i + (2t - 1)\mathbf{V}^\ddagger$ if $1/2 < t \leq 1$ and denote $\mathcal{V}_t := (\mathbf{V}_t^1, \ldots, \mathbf{V}_t^n) \in (C_0^1(\mathbf{R}^3)^3)^n$. Eventually, a continuous function linking $c^\dagger$ to $c^\ddagger$ is given by $t \in [0, 1] \mapsto c_t \in \mathcal{C}(n)$ with $c_t := (\varrho^\dagger, \vartheta^\dagger, \mathcal{V}_{3t/2})$ if $0 \leq t \leq 1/3$, $c_t := (\varrho_{3t-1}, \vartheta_{3t-1}, \mathcal{V}_{1/2})$ if $1/3 < t \leq 2/3$ and $c_t := (\varrho^\ddagger, \vartheta^\ddagger, \mathcal{V}_{3t/2-1/2})$ if $2/3 < t \leq 1$. ☐

We omit the proof of the following corollary, similar to that of the theorem above:

Corollary 2.4. *For any positive integer n, the set $\mathcal{C}_X(n)$ is an analytic connected embedded submanifold of $C^0(\bar B) \times C_0^1(\mathbf{R}^3)^3 \times (C_0^1(\mathbf{R}^3)^3)^n \times \mathbf{R}^n$ of codimension $N := 3(n + 2)(n + 1)/2$.*

We denote π the projection of $\mathcal{C}(n)$ onto $C^0(\bar B) \times D_0^1(\mathbf{R}^3)$ defined by $\pi(c) = (\varrho, \vartheta)$ for all $c := (\varrho, \vartheta, \mathcal{V}) \in \mathcal{C}(n)$. The proof of the following corollary is a straightforward consequence of arguments already used in the proof of Theorem 2.3:

Corollary 2.5. *For any positive integer n and for any $(\varrho, \vartheta) \in \pi(\mathcal{C}(n))$, the section $\pi^{-1}(\{(\varrho, \vartheta)\})$ is an embedded connected analytic submanifold of $\{\varrho\} \times \{\vartheta\} \times (C_0^1(\mathbf{R}^3)^3)^n$ (identified with $(C_0^1(\mathbf{R}^3)^3)^n$) of codimension $3n(n + 3)/2$.*

3. Sensitivity Analysis of the Mass Matrices. For any positive integers k and p, we denote $\mathrm{M}(k, p)$ the vector space of the matrices of size $k \times p$ (or simply $\mathrm{M}(k)$ when $k = p$).

Theorem 3.1. *For any positive integer n, the mappings $\mathbf{c} \in \mathcal{C}_X(n) \mapsto \mathbb{M}^r(\mathbf{c}) \in \mathrm{M}(6)$ and $\mathbf{c} \in \mathcal{C}_X(n) \mapsto \mathbb{N}(\mathbf{c}) \in \mathrm{M}(6, n)$ are analytic.*

The method followed in this proof is inspired from [6]. The result already appeared in [13], in a slightly different form though. Due to its crucial importance for our purpose, we recall here the

main ideas.

Let us begin with a preliminary lemma of which the statement requires introducing some material. Thus, we denote $F := \mathbf{R}^3 \setminus \bar{B}$ (recall that B is the unit ball, $\Sigma := \partial B$ and $\mathbf{n}$ is the unit normal to Σ directed toward the interior of B). For all $\xi \in D^1_0(\mathbf{R}^3)$, we set $\varXi := \mathrm{Id} + \xi$, $\mathcal{B}_\xi := \varXi(B)$, $\mathcal{F}_\xi := \varXi(F)$, $\Sigma_\xi := \varXi(\Sigma)$ and $\mathbf{n}_\xi$ stands for the unit normal to Σ_ξ directed toward the interior of $\mathcal{B}_\xi$. We denote $\mathbf{q} := (\xi, \mathcal{W})$, with $\mathcal{W} := (\mathbf{W}^1, \mathbf{W}^2) \subset (C^1_0(\mathbf{R}^3)^3)^2$, the elements of $\mathcal{Q} := D^1_0(\mathbf{R}^3) \times (C^1_0(\mathbf{R}^3)^3)^2$ and $\mathbf{w}^i_\xi := \mathbf{W}^i(\varXi^{-1})$ $(i = 1, 2)$. Finally, for every $\mathbf{q} \in \mathcal{Q}$, we define:

$$\varPhi(\mathbf{q}) := \int_{\mathcal{F}_\xi} \nabla \psi^1_{\mathbf{q}}(x) \cdot \nabla \psi^2_{\mathbf{q}}(x)\, \mathrm{d}x, \tag{3.1}$$

where, for every $i = 1, 2$, the function $\psi^i_{\mathbf{q}} \in W^1(\mathcal{F}_\xi)$ (recall that the function spaces are defined in Section A) is solution to the Laplace equation $-\Delta \psi^i_{\mathbf{q}} = 0$ in $\mathcal{F}_\xi$ with Neumann boundary data $\partial_n \psi^i_{\mathbf{q}} = \mathbf{w}^i_\xi \cdot \mathbf{n}_\xi$ on Σ_ξ. The solution has to be understood in the weak sense, namely:

$$\int_{\mathcal{F}_\xi} \nabla \psi^i_{\mathbf{q}}(x) \cdot \nabla \varphi(x)\, \mathrm{d}x = \int_{\Sigma_\xi} (\mathbf{w}^i_\xi \cdot \mathbf{n}_\xi)(x) \varphi(x)\, \mathrm{d}\sigma, \quad \forall\, \varphi \in W^1(\mathcal{F}_\xi). \tag{3.2}$$

Lemma 3.2. *The mapping $\mathbf{q} \in \mathcal{Q} \mapsto \varPhi(\mathbf{q}) \in \mathbf{R}$ is analytic.*

Proof. We pull back relation (3.2) onto the domain F using the diffeomorphism $\varXi$. We get:

$$\int_F \nabla \varPsi^i_{\mathbf{q}}(x) \mathbb{A}_\xi(x) \cdot \nabla(\varphi \circ \varXi)(x)\, \mathrm{d}x = \int_\Sigma (\mathbf{W}^i(x) \cdot \mathbf{n}_\xi(\varXi(x)) \varphi(\varXi(x)) J^\sigma_\xi(x)\, \mathrm{d}\sigma, \quad \forall\, \varphi \in W^1(\mathcal{F}_\xi),$$

where $\varPsi^i_{\mathbf{q}} := \psi^i_{\mathbf{q}} \circ \varXi$, $J_\xi := \det(\nabla \varXi)$, $\mathbb{A}_\xi := (\nabla \varXi^t \nabla \varXi)^{-1} J_\xi$ and $J^\sigma_\xi := \|(\nabla \varXi^{-1})^t \mathbf{n}\|_{\mathbf{R}^3} J_\xi$ (usually referred to as the tangential Jacobian). In (3.2), if we specialize the test function to have the form $\varphi := \phi \circ \varXi^{-1}$ with $\phi \in W^1(F)$, we obtain $\int_F \varPsi^i_{\mathbf{q}}(x) \mathbb{A}_\xi(x) \cdot \nabla \phi(x)\, \mathrm{d}x = \int_\Sigma b^i_{\mathbf{q}}(x) \phi(x)\, \mathrm{d}\sigma$ for all $\phi \in W^1(F)$, where $b^i_{\mathbf{q}} := (\mathbf{W}^i \cdot \mathbf{n}_\xi(\varXi)) J^\sigma_\xi$ $(i = 1, 2)$. We now claim that the mapping $\xi \in D^1_0(\mathbf{R}^3) \mapsto \mathbb{A}_\xi - \mathrm{Id} \in E^0_0(\Omega, \mathrm{M}(3))$ is analytic. Indeed, the mappings $\xi \in D^1_0(\mathbf{R}^3) \mapsto \nabla \varXi^t \nabla \varXi - \mathrm{Id} \in E^0_0(\Omega, \mathrm{M}(3))$, $A \in E^0_0(\Omega, \mathrm{M}(3)) \mapsto (\mathrm{Id}+A)^{-1} - \mathrm{Id} \in E^0_0(\Omega, \mathrm{M}(3))$ and $\xi \in D^1_0(\mathbf{R}^3) \mapsto J_\xi - 1 \in C^0_0(\mathbf{R}^3)$ are analytic. Reasoning the same way, we can show that the mapping $\xi \in D^1_0(\mathbf{R}^3) \mapsto J^\sigma_\xi \in C^0(\Sigma)$ is analytic as well (notice that for all $\xi \in D^1_0(\mathbf{R}^3)$, the function $(\nabla \varXi^{-1})^t \mathbf{n}$ never vanishes on Σ). It is more complicated to prove that $\xi \in D^1_0(\mathbf{R}^3) \mapsto \mathbf{n}_\xi \circ \varXi \in C^0(\Sigma)^2$ is analytic, so we refer to [13] for the details. This last result entails the analyticity of $\mathbf{q} \in \mathcal{Q} \mapsto b^i_{\mathbf{q}} \in C^0(\Sigma)$. Then, denoting by $W^1(F)'$ the dual space to $W^1(F)$, we consider the mapping $\varGamma : (\mathbf{q}, u) \in \mathcal{Q} \times W^1(F) \mapsto \langle \mathbb{A}_\xi, u, \cdot \rangle - \langle b^i_{\mathbf{q}}, \cdot \rangle \in W^1(F)'$, where $\langle \mathbb{A}_\xi, u, \phi \rangle := \int_F \nabla u(x) \mathbb{A}_\xi(x) \cdot \nabla \phi(x)\, \mathrm{d}x$ and $\langle b^i_{\mathbf{q}}, \phi \rangle := \int_\Sigma b^i_{\mathbf{q}}(x) \phi(x)\, \mathrm{d}\sigma$ $(\phi \in W^1(F))$. We wish now to apply the implicit function theorem (analytic version in Banach spaces, as stated in [17]) to the analytic function $\varGamma$. Observe, though, that we are only interested in the regularity result and not in the existence and uniqueness. Indeed, we already know that for every $\mathbf{q} \in \mathcal{Q}$, there exists a unique function $\varPsi^i_{\mathbf{q}} \in W^1(F)$ such that $\varGamma(\mathbf{q}, \varPsi^i_{\mathbf{q}}) = 0$. The function $\varPsi^i_{\mathbf{q}}$ is equal to $\psi^i_{\mathbf{q}} \circ \varXi$ where $\psi^i_{\mathbf{q}}$ is the unique solution to the well posed variational problem (3.2). The partial derivative $\partial_u \varGamma(\mathbf{q}, \varPsi^i_{\mathbf{q}})$ is defined by:

$$\langle \partial_u \varGamma(\mathbf{q}, \varPsi^i_{\mathbf{q}}), u, \phi \rangle = \int_F \nabla u(x) \mathbb{A}_\xi(x) \cdot \nabla \phi(x)\, \mathrm{d}x, \quad \forall \phi \in W^1(F). \tag{3.3}$$

For all $\xi \in D^1_0(\mathbf{R}^3)$, the matrix $\mathbb{A}_\xi$ is uniformly elliptic in $\mathbf{R}^3$ (there exists $\alpha_\xi > 0$ such that $X^t \mathbb{A}_\xi(x) X > \alpha_\xi \|X\|^2_{\mathbf{R}^3}$ for all $X \in \mathbf{R}^3$ and all $x \in \mathbf{R}^3$). We deduce that the right hand side of

(3.3) can be chosen as the scalar product in $W^1(F)$ and hence that $\partial_u\Gamma(\mathbf{q},\Psi^i_{\mathbf{q}})$ is a continuous isomorphism from $W^1(F)$ onto its dual space according to the Riesz representation theorem. The implicit function theorem applies and asserts that the mapping $\mathbf{q}\in\mathcal{Q}\mapsto\Psi^i_{\mathbf{q}}\in W^1(F)$ is analytic.

To conclude the proof, it remains only to observe that the function $\Phi(\mathbf{q})$ introduced in (3.1) can be rewritten, upon a change of variables as $\Phi(\mathbf{q})=\int_F\nabla\Psi^1_{\mathbf{q}}(x)\mathbb{A}_\xi(x)\cdot\nabla\Psi^2_{\mathbf{q}}(x)\,\mathrm{d}x$, which is indeed analytic as a composition of analytic functions. ∎

We can now give the proof of Theorem 3.1.

Proof. Recall that the elements of the matrix $\mathbb{M}^r_f(\mathbf{c})$ are defined in (1.4) and those of $\mathbb{N}(\mathbf{c})$ in (1.5). For any $\mathbf{c}:=(c,s)\in\mathcal{C}_X(n)$, where $c:=(\varrho,\vartheta,\mathcal{V})$, we apply the lemma with $\xi:=\vartheta+\sum_{i=1}^n s_i\mathbf{V}_i$ and $\mathbf{W}^1,\mathbf{W}^2\in\{\mathbf{e}_i\times\Xi,\,\mathbf{e}_i,\,i=1,2,3\}$ to get that the mapping $\mathbf{c}\in\mathcal{C}_X(n)\mapsto\mathbb{M}^r_f(\mathbf{c})\in\mathrm{M}(6)$ is analytic. To prove the analyticity of the elements of $\mathbb{N}(\mathbf{c})$, we apply the lemma again with $\xi:=\vartheta+\sum_{i=1}^n s_i\mathbf{V}_i$, $\mathbf{W}^1\in\{\mathbf{e}_i\times\Xi,\,\mathbf{e}_i,\,i=1,2,3\}$ and $\mathbf{W}^2\in\{\mathbf{V}_1,\dots,\mathbf{V}_n\}$. Eventually, the analyticity of the elements of $\mathbb{M}^r_b(\mathbf{c})$ is straightforward after rewriting the inertia tensor $\mathbb{I}(\mathbf{c})$ defined in (1.3) in the form (upon a change of variables) $\mathbb{I}(\mathbf{c})=\int_B\varrho[\|\Xi\|^2_{\mathbf{R}^3}\mathrm{Id}-\Xi\otimes\Xi]\,\mathrm{d}x$, still with $\Xi:=\mathrm{Id}+\xi$ and $\xi:=\vartheta+\sum_{i=1}^n s_i\mathbf{V}_i$. ∎

4. Control.

4.1. Controllable SC. Let us fix $c\in\mathcal{C}(n)$ (for some positive integer n) and recall that $\mathcal{S}(c)$ is the connected open subspace of $\mathbf{R}^n$ such that $(c,s)\in\mathcal{C}_X(n)$. Introducing $(\mathbf{f}_1,\dots,\mathbf{f}_n)$ an ordered orthonormal basis of $\mathbf{R}^n$, we can seek the function $t\in[0,T]\mapsto s(t)\in\mathcal{S}(c)$ as the solution of the ODE $\dot s(t)=\sum_{i=1}^n\lambda_i(t)\mathbf{f}_i$ where the functions $\lambda_i:t\in[0,T]\mapsto\lambda_i(t)\in\mathbf{R}$ are the new controls, and rewrite once more the dynamics (2.1) as:

$$\begin{pmatrix}\mathbf{\Omega}\\ \mathbf{v}\\ \dot s\end{pmatrix}=\sum_{i=1}^n\lambda_i(t)\begin{pmatrix}-\mathbb{M}^r(c,s)^{-1}\mathbb{N}(c,s)\mathbf{f}_i\\ \mathbf{f}_i\end{pmatrix},\qquad(t\geq 0).\tag{4.1}$$

It is worth remarking that in this form, s is no more the control but a variable which is meant to be controlled and $c\in\mathcal{C}(n)$ is a parameter of the dynamics. Considering (4.1), we are quite naturally led to introduce, for all $\mathbf{c}\in\mathcal{C}_X(n)$, the vector fields $\mathbf{X}_i(\mathbf{c}):=-\mathbb{M}^r(\mathbf{c})^{-1}\mathbb{N}(\mathbf{c})\mathbf{f}_i\in\mathbf{R}^6$, $\mathbf{Y}_i(\mathbf{c}):=(\hat{\mathbf{X}}^1_i(\mathbf{c}),\mathbf{X}^2_i(\mathbf{c}),\mathbf{f}_i)^t\in T_{\mathrm{Id}}\mathrm{SO}(3)\times\mathbf{R}^3\times\mathbf{R}^n$ (we have used here the notation $\mathbf{X}_i:=(\mathbf{X}^1_i,\mathbf{X}^2_i)^t\in\mathbf{R}^3\times\mathbf{R}^3$) and $\mathbf{Z}^i_c(R,s):=\mathcal{R}_R\mathbf{Y}_i(\mathbf{c})\in T_R\mathrm{SO}(3)\times\mathbf{R}^3\times\mathbf{R}^n$ where $\mathcal{R}_R:=\mathrm{diag}(R,R,\mathrm{Id})\in\mathrm{SO}(6+n)$ is a bloc diagonal matrix. The dynamics (4.1) and the ODE (1.6b) can be gathered into a unique differential system:

$$\frac{d}{dt}\begin{pmatrix}R\\ \mathbf{r}\\ s\end{pmatrix}=\sum_{i=1}^n\lambda_i(t)\mathbf{Z}^i_c(R,s),\qquad(t\geq 0).\tag{4.2}$$

For every $i=1,\dots,n$, the function $(R,\mathbf{r},s)\in\mathrm{SO}(3)\times\mathbf{R}^3\times\mathcal{S}(c)\mapsto\mathbf{Z}^i_c(R,s)\in T_R\mathrm{SO}(3)\times\mathbf{R}^3\times\mathbf{R}^n$ can be seen as an analytic vector field (constant in $\mathbf{r}$) on the analytic connected manifold $\mathcal{M}(c):=\mathrm{SO}(3)\times\mathbf{R}^3\times\mathcal{S}(c)$. We denote ζ any element $(R,\mathbf{r},s)\in\mathcal{M}(c)$ and we define $\mathcal{Z}(c)$ as the family of vector fields $(\mathbf{Z}^i_c)_{1\leq i\leq n}$ on $\mathcal{M}(c)$.

LEMMA 4.1. *Let c be a SC fixed in $\mathcal{C}(n)$ (n a positive integer). If there exists $\zeta\in\mathcal{M}(c)$ such that $\dim\mathrm{Lie}_\zeta\mathcal{Z}(c)=6+n$, then the orbit of $\mathcal{Z}(c)$ through any $\zeta\in\mathcal{M}(c)$ is equal to the whole manifold $\mathcal{M}(c)$.*

Proof. Rashevsky Chow Theorem (see [1]) applies: If $\mathrm{Lie}_\zeta\mathcal{Z}(c)=T_\zeta\mathcal{M}(c)$ for all $\zeta\in\mathcal{M}(c)$ (or more precisely, for all $(R,s)\in\mathrm{SO}(3)\times\mathcal{S}(c)$ since $\mathbf{Z}^i_c$ does not depend on $\mathbf{r}$) then the orbit of

$\mathcal{Z}(c)$ through any point of $\mathcal{M}(c)$ is equal to the whole manifold. Let us compute the Lie bracket $[\mathbf{Z}_c^i(R,s),\,\mathbf{Z}_c^j(R,s)]$ for $1\leq i,j\leq n$ and $(R,s)\in\mathrm{SO}(3)\times\mathcal{S}(c)$. We get:

$$[\mathbf{Z}_c^i(R,s),\,\mathbf{Z}_c^j(R,s)]=\mathcal{R}_R\begin{pmatrix}(\widehat{\mathbf{X}_i^1\times\mathbf{X}_j^1})(\mathbf{c})\\(\mathbf{X}_i^1\times\mathbf{X}_j^2-\mathbf{X}_j^1\times\mathbf{X}_i^2)(\mathbf{c})\\\mathbf{0}\end{pmatrix}+\mathcal{R}_R\begin{pmatrix}(\widehat{\partial_{s_i}\mathbf{X}_j^1-\partial_{s_j}\mathbf{X}_i^1})(\mathbf{c})\\(\partial_{s_i}\mathbf{X}_j^2-\partial_{s_j}\mathbf{X}_i^2)(\mathbf{c})\\\mathbf{0}\end{pmatrix}. \qquad (4.3)$$

By induction, we can similarly prove that the Lie brackets of any order at any point $\zeta\in\mathcal{M}(c)$ have the same general form, namely the matrix $\mathcal{R}_R$ multiplied by an element of $T_{(\mathrm{Id},\mathbf{0},s)}\mathcal{M}(c)$. We deduce that the dimension of the Lie algebra at any point of $\mathcal{M}(c)$ depends only on s. According to the Orbit Theorem (see [1]), the dimension of the Lie algebra is constant along any orbit. But according to the particular form of the vector fields $\mathbf{Z}_c^i$ (whose last n components form a basis of $\mathbf{R}^n$), the projection of any orbit on $\mathcal{S}(c)$ turns out to be the whole set $\mathcal{S}(c)$ (or, in other words, for any $s\in\mathcal{S}(c)$ and for any orbit, there is a point of the orbit for which the last component is s). Assume now that $\dim\mathrm{Lie}_{\zeta^*}\mathcal{Z}(c)=6+n$ at some particular point $\zeta^*:=(R^*,\mathbf{r}^*,s^*)\in\mathcal{M}(c)$. Then, according to the Orbit Theorem, for any $s\in\mathcal{S}(c)$, there exists at least one point $(R_s,\mathbf{r}_s,s)\in\mathcal{M}(c)$ such that $\dim\mathrm{Lie}_{(R_s,\mathbf{r}_s,s)}\mathcal{Z}(c)=6+n$. But since the dimension of the Lie algebra does not depend on the variables R and $\mathbf{r}$, we conclude that $\dim\mathrm{Lie}_\zeta\mathcal{Z}(c)=6+n$ for all $\zeta\in\mathcal{M}(c)$. ☐

Definition 4.2. *We say that c, a SC in $C(n)$ (for some integer n) is controllable if there exists $\zeta\in\mathcal{M}(c)$ such that* $\dim\mathrm{Lie}_\zeta\mathcal{Z}(c)=6+n$.

It is obvious that for a SC to be controllable, the integer n has to be larger or equal to 2. The following result is quite classical (a proof can be found in [4]):

Proposition 4.3. *Let $c\in C(n)$ (for some integer n) be controllable (with the usual notation $c:=(\varrho,\vartheta,\mathcal{V})$, $\mathcal{V}:=(\mathbf{V}_1,\ldots,\mathbf{V}_n)$ and $\vartheta_s:=\vartheta+\sum_{i=1}^n s_i\mathbf{V}_i$ for every $s\in\mathcal{S}(c)$). Then for any given continuous function $t\in[0,T]\mapsto(\bar{R}(t),\bar{\mathbf{r}}(t),\bar{s}(t))\in\mathrm{SO}(3)\times\mathbf{R}^3\times\mathcal{S}(c)$ and for any $\varepsilon>0$, there exist n C^1 functions $\lambda_i:[0,T]\to\mathbf{R}$ $(i=1,\ldots,n)$ such that:*

1. $\sup_{t\in[0,T]}\Big(\|\bar{R}(t)-R(t)\|_{\mathrm{M}(3)}+\|\bar{\mathbf{r}}(t)-\mathbf{r}(t)\|_{\mathbf{R}^3}+\|\vartheta_{\bar{s}(t)}-\vartheta_{s(t)}\|_{C^1_0(\mathbf{R}^3)^3}\Big)<\varepsilon$;
2. $R(T)=\bar{R}(T)$, $\mathbf{r}(T)=\bar{\mathbf{r}}(T)$ *and* $s(T)=\bar{s}(T)$;

where $t\in[0,T]\mapsto(R(t),\mathbf{r}(t),s(t))\in\mathcal{M}(c)$ is the unique solution to the ODE (4.2) *with Cauchy data $R(0)=\bar{R}(0)\in\mathrm{SO}(3)$, $\mathbf{r}(0)=\bar{\mathbf{r}}(0)\in\mathbf{R}^3$, $s(0)=\bar{s}(0)\in\mathcal{S}(c)$.*

Let us mention some other quite elementary properties that will be used later on:

Proposition 4.4.

1. *If $c:=(\varrho,\vartheta,\mathcal{V})\in\mathcal{C}(n)$ $(n\geq 2)$ is a controllable SC with $\mathcal{V}:=(\mathbf{V}_1,\ldots,\mathbf{V}_n)\in(C^1_0(\mathbf{R}^3)^3)^n$ then any $c^+:=(\varrho,\vartheta,\mathcal{V}^+)\in\mathcal{C}(n+1)$ such that $\mathcal{V}^+:=(\mathbf{V}_1,\ldots,\mathbf{V}_n,\mathbf{V}_{n+1})\in(C^1_0(\mathbf{R}^3)^3)^{n+1}$ (for some $\mathbf{V}_{n+1}\in C^m_0(\mathbf{R}^3)^3$) is a controllable SC as well.*
2. *If $c:=(\varrho,\vartheta,\mathcal{V})\in\mathcal{C}(n)$ $(n\geq 2)$ is a controllable SC, then for any $\vartheta^*\in\{\vartheta+\sum_{i=1}^n s_i\mathbf{V}_i,\ s\in\mathcal{S}(c)\}$ the element $c^*:=(\varrho,\vartheta^*,\mathcal{V})$ belongs to $\mathcal{C}(n)$ and is a controllable SC as well.*
3. *If $c:=(\varrho,\vartheta,\mathcal{V})\in\mathcal{C}(n)$ $(n\geq 2)$ is a controllable SC, then all of the controllable SC $c^*:=(\varrho,\vartheta,\mathcal{V}^*)$ in $\mathcal{C}(n)$ ($\mathcal{V}^*\in(C^1_0(\mathbf{R}^3)^3)^n$) form an open dense subset of the section $\pi^{-1}(\{(\varrho,\vartheta)\})$ (for the induced topology).*
4. *If there exists a SC in $\mathcal{C}(n)$ for some $n\geq 2$ then, for any $k\geq n$, all of the controllable SC in $\mathcal{C}(k)$ form an open dense subset of $\mathcal{C}(k)$ (for the induced topology).*

Proof. The two first assertions are obvious so let us address directly the third point. Denote $\mathcal{E}_k$ (k positive integer) the set of all of the vectors fields on $\mathcal{M}(c)$ obtained as Lie brackets of order lower or equal to k from elements of $\mathcal{Z}(c)$. Then, consider the determinants of all of the different families of $6+n$ elements of $\mathcal{E}_k$ as analytic functions in the variable $\mathcal{V}$ (the other variables ϱ, ϑ and

$s=0$ being fixed). Since c is controllable, there exist at least one k and one family of $6+n$ elements in $\mathcal{E}_k$ whose determinant is nonzero. According to Corollary 2.5 and basic properties of analytic functions (see [17]), the determinant may vanish only in a closed subset with empty interior of the section $\pi^{-1}(\{(\varrho,\vartheta)\})$ (for the induced topology). The proof of the last point is similar. $\square$

4.2. Building a controllable SC. In this subsection, we are interested in computing the Lie brackets of first order $[\mathbf{Z}_c^i(R,s),\,\mathbf{Z}_c^j(R,s)]$ at $(R,s)=(\mathrm{Id},0)$, for a particular SC.

General computations. Starting from identity (4.3) and focusing on the second term in the right hand side, we have, for all $c\in\mathcal{C}(n)$ $(n\geq 2)$, all $s\in\mathcal{S}(c)$ and all $i,j=1,\ldots,n$:

$$\begin{aligned}\partial_{s_i}\mathbf{X}_j(\mathbf{c})-\partial_{s_j}\mathbf{X}_i(\mathbf{c})=\mathbb{M}^r(\mathbf{c})^{-1}\Big[&(\partial_{s_j}\mathbb{M}^r(\mathbf{c})\mathbf{X}_i(\mathbf{c})-\partial_{s_i}\mathbb{M}^r(\mathbf{c})\mathbf{X}_j(\mathbf{c}))\\&+(\partial_{s_j}\mathbb{N}(\mathbf{c})\mathbf{f}_i-\partial_{s_i}\mathbb{N}(\mathbf{c})\mathbf{f}_j)\Big].\end{aligned}\tag{4.4}$$

From the decomposition $\mathbb{M}^r(\mathbf{c}):=\mathbb{M}^r_b(\mathbf{c})+\mathbb{M}^r_f(\mathbf{c})$, we deduce that:

$$\begin{aligned}&\partial_{s_j}\mathbb{M}^r(\mathbf{c})\mathbf{X}_i(\mathbf{c})-\partial_{s_i}\mathbb{M}^r(\mathbf{c})\mathbf{X}_j(\mathbf{c})=\\&\qquad(\partial_{s_j}\mathbb{M}^r_b(\mathbf{c})\mathbf{X}_i(\mathbf{c})-\partial_{s_i}\mathbb{M}^r_b(\mathbf{c})\mathbf{X}_j(\mathbf{c}))+(\partial_{s_j}\mathbb{M}^r_f(\mathbf{c})\mathbf{X}_i(\mathbf{c})-\partial_{s_i}\mathbb{M}^r_f(\mathbf{c})\mathbf{X}_j(\mathbf{c})).\end{aligned}\tag{4.5}$$

We can easily compute the first term in the right hand side when $s=0$. Thus, for all $i,j=1,\ldots,n$, we have:

$$[\partial_{s_j}\mathbb{M}^r_b(\mathbf{c})\mathbf{X}_i(\mathbf{c})-\partial_{s_i}\mathbb{M}^r_b(\mathbf{c})\mathbf{X}_j(\mathbf{c})]_{s=0}=\eta_j\begin{bmatrix}\mathrm{Id}&0\\0&0\end{bmatrix}\mathbf{X}_i(\mathbf{c})|_{s=0}-\eta_i\begin{bmatrix}\mathrm{Id}&0\\0&0\end{bmatrix}\mathbf{X}_j(\mathbf{c})|_{s=0},\tag{4.6}$$

where $\eta_i:=2\int_B\varrho(x)\,\mathbf{V}_i(x)\cdot\Theta(x)\,\mathrm{d}x$.

Rigid shell's deformation. We consider now shape changes that reduce to rigid displacements on the swimmer's boundary Σ (but obviously not inside the body for the self-propelled constraints not to be violated). The idea of using such deformations stemmed from the reading of the article [9]. However, rigid deformations of the shell can not be considered within the modeling described in Section 2 (because rigid deformations do not fit with the general form of the diffeomorphisms described in Section 2). Let us explain how to deal with this difficulty.

Let $c:=(\varrho,\vartheta,\mathcal{V})\in\mathcal{C}(6)$ be given such that $\mathcal{V}:=(\mathbf{V}_1,\ldots,\mathbf{V}_6)$ with $\mathbf{V}_i|_\Sigma(x):=\mathbf{e}_i\times(x+\vartheta(x))$ $(i=1,2,3)$ and $\mathbf{V}_i|_\Sigma(x):=\mathbf{e}_{i-3}$ $(i=4,5,6)$ (Proposition C.1 in the appendix guarantees the existence of such vector fields satisfying furthermore $\eta_i=0$ for all $i=1,\ldots,6$). Thus, the shape changes we are considering read $\Theta_s:=\mathrm{Id}+\vartheta+\sum_{i=1}^6 s_i\mathbf{V}_i$ $(s\in\mathcal{S}(c))$. Let us define also $\Xi_s:=R_s(\mathrm{Id}+\vartheta)+\boldsymbol{\tau}_s$ where $R_s:=\exp(s_1\hat{\mathbf{e}}_1)\exp(s_2\hat{\mathbf{e}}_2)\exp(s_3\hat{\mathbf{e}}_3)\in\mathrm{SO}(3)$ and $\boldsymbol{\tau}_s:=\sum_{i=1}^3 s_{i+3}\mathbf{e}_i\in\mathbf{R}^3$. Thus, Θ_s is a diffeomorphism which can be achieved within our modeling while Ξ_s is a *true* rigid deformation.

In order to determine the expressions of the terms $\partial_{s_i}\mathbb{M}^r_f(\mathbf{c})|_{s=0}$ $(i=1,\ldots,6)$ arising in the computation of the Lie brackets, we have to compare (using the notation of Lemma 3.2) $\partial_{s_i}\Phi(\mathbf{q}^1_s)|_{s=0}$ and $\partial_{s_i}\Phi(\mathbf{q}^2_s)|_{s=0}$ $(i=1,\ldots,6)$ where:

- $\mathbf{q}^1_s:=(\xi^1_s,\mathcal{W}^1_s)$ with $\xi^1_s:=\Xi_s-\mathrm{Id}$, $\mathcal{W}^1_s:=(\mathbf{W}^{1,1}_s,\mathbf{W}^{1,2}_s)$ and $\mathbf{W}^{1,1}_s$, $\mathbf{W}^{1,2}_s\in\{\mathbf{e}_i\times\Xi_s,\,\mathbf{e}_i,\,i=1,2,3\}$ (settings corresponding to a true rigid deformation of the shell);
- $\mathbf{q}^2_s:=(\xi^2_s,\mathcal{W}^2_s)$ with $\xi^2_s:=\Theta_s-\mathrm{Id}=\vartheta+\sum_{i=1}^n s_i\mathbf{V}_i$, $\mathcal{W}^2_s:=(\mathbf{W}^{2,1}_s,\mathbf{W}^{2,2}_s)$ and $\mathbf{W}^{2,1}_s,\mathbf{W}^{2,2}_s\in\{\mathbf{e}_i\times\Theta_s,\,\mathbf{e}_i,\,i=1,2,3\}$ (settings allowed in our modeling).

Remark that $\partial_{s_i}\Theta_s|_{s=0} = \partial_{s_i}\Xi_s|_{s=0}$ on Σ for all $i = 1, \dots, 6$, so the deformations are tangent at $s = 0$. It entails that $\partial_{s_i}\mathbf{q}^1_s|_{s=0} = \partial_{s_i}\mathbf{q}^2_s|_{s=0}$. Since, in addition, $\mathbf{q}^1_{s=0} = \mathbf{q}^2_{s=0}$ and $\partial_{s_i}\Phi(\mathbf{q}^k_s)|_{s=0} = \langle \partial_{\mathbf{q}}\Phi(\mathbf{q}^k_s), \partial_{s_i}\mathbf{q}^k_s\rangle|_{s=0}$ for $k = 1, 2$ and $i = 1, \dots, 6$, we deduce that $\partial_{s_i}\Phi(\mathbf{q}^1_s)|_{s=0} = \partial_{s_i}\Phi(\mathbf{q}^2_s)|_{s=0}$ for all $i = 1, \dots, 6$.

We apply the diffeomorphism $\Xi_s := R_s(\mathrm{Id}+\vartheta)+\boldsymbol{\tau}_s$ to $\bar{B}$ to obtain the domain of the deformed swimmer. We denote $\mathcal{B}^\diamond := (\mathrm{Id}+\vartheta)(B)$ (which can be seen as the shape of the swimmer at rest, i.e. when $s = 0$), $\Sigma^\diamond := \partial\mathcal{B}^\diamond$, $\mathcal{F}^\diamond := \mathbf{R}^3 \setminus \bar{\mathcal{B}}^\diamond$ and $\mathcal{B}_s := \Xi_s(B)$, $\mathcal{F}_s := \mathbf{R}^3 \setminus \bar{\mathcal{B}}_s$, $\Sigma_s := \partial\mathcal{B}_s$. We seek the potential $\psi^1_{\mathbf{c}}$, defined and harmonic in $\mathcal{F}_s$ in the form $\psi^1_{\mathbf{c}}(x) = \tilde{\psi}^1_c(R^t_s(x - \boldsymbol{\tau}_s))$ where the function $\tilde{\psi}^1_c$ defined on $\mathcal{F}^\diamond$ has to be determined. It is obvious that $\tilde{\psi}^1_c$ is harmonic in $\mathcal{F}^\diamond$ and we have only to determine the boundary conditions on $\Sigma^\diamond$. For all $y \in \Sigma^\diamond$, we denote $x := R_s y + \boldsymbol{\tau}_s \in \Sigma_s$. The relation $\mathbf{n}|_{\Sigma_s}(R_s y+\boldsymbol{\tau}_s) = R_s\mathbf{n}|_{\Sigma^\diamond}(y)$ entails that $\nabla\tilde{\psi}^1_c(y)\cdot\mathbf{n}|_{\Sigma^\diamond}(y) = \nabla\psi^1_{\mathbf{c}}(R_s y+\boldsymbol{\tau}_s)\cdot\mathbf{n}|_{\Sigma_s}(R_s y+\boldsymbol{\tau}_s) = \nabla\psi^1_{\mathbf{c}}(x)\cdot\mathbf{n}(x)$, and this quantity has to be equal to $(\mathbf{e}_1\times x)\cdot\mathbf{n}(x)$. We deduce after some elementary algebra that $\partial_n\tilde{\psi}^1_c(y) = (R^t_s\mathbf{e}_1 \times y)\cdot\mathbf{n}(y) + (R^t_s\mathbf{e}_1 \times R^t_s\boldsymbol{\tau}_s)\cdot\mathbf{n}(y)$. Proceeding similarly and with obvious notation, we obtain more generally that:

$$\begin{aligned}
&\partial_n\tilde{\psi}^i_c(y) = (R^t_s\mathbf{e}_i \times y)\cdot\mathbf{n}(y) + (R^t_s\mathbf{e}_i \times R^t_s\boldsymbol{\tau}_s)\cdot\mathbf{n}(y), && (i = 1, 2, 3),\\
\text{and}\qquad &\partial_n\tilde{\psi}^i_c(y) = R^t_s\mathbf{e}_{i-3}\cdot\mathbf{n}(y), && (i = 4, 5, 6).
\end{aligned}$$

Denoting $\mathbb{M}^\diamond_f := \mathbb{M}^r_f(\mathbf{c})|_{s=0}$, we get the relations:

$$\mathbb{M}^r_f(c, s) = \begin{bmatrix}\mathrm{Id} & \hat{\boldsymbol{\tau}}_s\\ 0 & \mathrm{Id}\end{bmatrix}\begin{bmatrix}R_s & 0\\ 0 & R_s\end{bmatrix}\mathbb{M}^\diamond_f\begin{bmatrix}R^t_s & 0\\ 0 & R^t_s\end{bmatrix}\begin{bmatrix}\mathrm{Id} & 0\\ -\hat{\boldsymbol{\tau}}_s & \mathrm{Id}\end{bmatrix},$$

and then, differentiating with respect to s_i:

$$\partial_{s_i}\mathbb{M}^r_f(c, s)|_{s=0} = \begin{bmatrix}\hat{\mathbf{e}}_i & 0\\ 0 & \hat{\mathbf{e}}_i\end{bmatrix}\mathbb{M}^\diamond_f - \mathbb{M}^\diamond_f\begin{bmatrix}\hat{\mathbf{e}}_i & 0\\ 0 & \hat{\mathbf{e}}_i\end{bmatrix}, \qquad (i = 1, 2, 3), \tag{4.7a}$$

$$\partial_{s_i}\mathbb{M}^r_f(c, s)|_{s=0} = \begin{bmatrix}0 & \hat{\mathbf{e}}_{i-3}\\ 0 & 0\end{bmatrix}\mathbb{M}^\diamond_f - \mathbb{M}^\diamond_f\begin{bmatrix}0 & 0\\ \hat{\mathbf{e}}_{i-3} & 0\end{bmatrix}, \qquad (i = 4, 5, 6). \tag{4.7b}$$

The reasoning is quite similar for the entries of the matrix $\mathbb{N}(\mathbf{c})$. We only have to be careful that the vector fields $\mathbf{W}^{2,2}_s$ can actually not depend on s (once more, this is due to our modeling). The elements of the matrix $\partial_{s_j}\mathbb{N}(\mathbf{c})|_{s=0}\mathbf{f}_i - \partial_{s_i}\mathbb{N}(\mathbf{c})|_{s=0}\mathbf{f}_j$ read (still with the notation of Lemma 3.2) $\partial_{s_j}\Phi(\mathbf{q}^i_s)|_{s=0} - \partial_{s_i}\Phi(\mathbf{q}^j_s)|_{s=0}$ $(i, j = 1, \dots, 6)$ with this time $\mathbf{q}^i_s := (\xi_s, \mathcal{W}^i_s)$, $\xi_s := \Xi_s - \mathrm{Id}$, $\mathcal{W}^i_s := (\mathbf{W}^1_s, \mathbf{W}^{i,2})$, $\mathbf{W}^1_s \in \{\mathbf{e}_k \times \Xi_s, \mathbf{e}_k, k = 1, 2, 3\}$ and $\mathbf{W}^{i,2} := \mathbf{e}_i \times (\mathrm{Id} + \vartheta)$ $(i = 1, 2, 3)$ or $\mathbf{W}^{i,2} = \mathbf{e}_{i-3}$ $(i = 4, 5, 6)$. The only difference with the elements of $\mathbb{M}^r_f(\mathbf{c})$ being that $\mathbf{W}^{i,2}$ does not depend on s for $i = 1, 2, 3$, we wish to reuse the preceding calculations. Invoking the chain rule, we have to subtract to $\partial_{s_j}\Phi(\mathbf{q}^i_s)|_{s=0} - \partial_{s_i}\Phi(\mathbf{q}^j_s)|_{s=0}$ the quantity $\langle\partial_{\mathbf{W}^2}\Phi(\mathbf{q}^i_s), \partial_{s_j}\mathbf{W}^{i,2}_s\rangle|_{s=0} - \langle\partial_{\mathbf{W}^2}\Phi(\mathbf{q}^j_s), \partial_{s_i}\mathbf{W}^{j,2}_s\rangle|_{s=0}$ which is quite easy to determine because Φ is linear in the variable $\mathbf{W}^2$. Thus, this last expression is merely equal to $\Phi(\mathbf{q}_{i,j})$ $(1 \le i, j \le 6)$ with $\mathbf{q}_{i,j} := (\vartheta, \mathcal{W}_{i,j})$, $\mathcal{W}_{i,j} := (\mathbf{W}^1, \mathbf{W}^2_{i,j})$, $\mathbf{W}^1 \in \{\mathbf{e}_k \times (\mathrm{Id} + \vartheta), \mathbf{e}_k, k = 1, 2, 3\}$ and

$$\mathbf{W}^2_{i,j} := \begin{cases}(\mathbf{e}_i \times \mathbf{e}_j) \times (\mathrm{Id} + \vartheta) & \text{if } 1 \le i, j \le 3,\\ (\mathbf{e}_i \times \mathbf{e}_j) & \text{if } 4 \le i \le 6\,, 1 \le j \le 3,\\ (\mathbf{e}_i \times \mathbf{e}_j) & \text{if } 1 \le i \le 3\,, 4 \le j \le 6,\\ \mathbf{0} & \text{if } 4 \le i, j \le 6.\end{cases}$$

We eventually obtain that:

$$\partial_{s_j}\mathbb{N}(\mathbf{c})|_{s=0}\mathbf{f}_i - \partial_{s_i}\mathbb{N}(\mathbf{c})|_{s=0}\mathbf{f}_j = \partial_{s_j}\mathbb{M}_f^r(\mathbf{c})|_{s=0}\mathbf{f}_i - \partial_{s_i}\mathbb{M}_f^r(\mathbf{c})|_{s=0}\mathbf{f}_j - \mathbb{N}^\diamond(\mathbf{f}_i \star \mathbf{f}_j), \tag{4.8}$$

where $\mathbb{N}^\diamond := \mathbb{N}(\mathbf{c})|_{s=0} = \mathbb{M}_f^\diamond$ and $\mathbf{f}_i \star \mathbf{f}_j := (\mathbf{f}_i^1 \times \mathbf{f}_j^1, \mathbf{f}_i^1 \times \mathbf{f}_j^2 - \mathbf{f}_j^1 \times \mathbf{f}_i^2)^t$ with the notation $\mathbf{f}_i = (\mathbf{f}_i^1, \mathbf{f}_i^2)^t \in \mathbf{R}^3 \times \mathbf{R}^3$.

Specifying the density and shape. The expression of the 6×6 symmetric added mass matrix $\mathbb{M}_f^\diamond$ depends only on the domain $\mathcal{F}^\diamond$ or equivalently on $\mathcal{B}^\diamond$. As stated in Proposition D.1 in the appendix, this matrix is positive definite if we choose ϑ in such a way that $\mathcal{B}^\diamond$ has no axis of symmetry. It entails that, up to a change of frame $\mathfrak{e}$ at the initial time, $\mathbb{M}_f^\diamond$ can be assumed to be diagonal with positive eigenvalues μ_j $(j = 1, \ldots, 6)$. On the other hand, denoting by $\mathrm{S}(3)^+$ the set of the 3×3 symmetric matrices that are positive definite, we can quite easily prove that for any $\mathcal{B}^\diamond$ (which means for any $\vartheta \in D_0^1(\mathbf{R}^3)$), the mapping $\varrho \in C^0(\bar{\mathcal{B}}^\diamond)^+ \mapsto \int_{\mathcal{B}^\diamond} \varrho(\|x\|_{\mathbf{R}^3}\mathrm{Id} - x \otimes x)\mathrm{d}x \in \mathrm{S}(3)^+$ is onto. We deduce that for any $(I_1, I_2, I_3) \in \mathbf{R}^3$ such that $I_i > 0$ for $i = 1, 2, 3$, there exists $\varrho \in C^0(\bar{B})^+$ such that the inertia tensor $\mathbb{I}(\mathbf{c})|_{s=0}$ is diagonal, equal to $\mathrm{diag}(I_1, I_2, I_3)$. In this case, the matrix $\mathbb{M}_b^r(\mathbf{c})|_{s=0}$ is diagonal as well, equal to $\mathrm{diag}(I_1, I_2, I_3, m, m, m,)$. We deduce that the vector fields $\mathbf{X}_i(\mathbf{c})|_{s=0}$ read $-\mu_i/(I_i + \mu_i)\mathbf{f}_i$ if $i = 1, 2, 3$ and $-\mu_i/(m + \mu_i)\mathbf{f}_i$ if $i = 4, 5, 6$. Summarizing (4.3), (4.4), (4.5), (4.6) (recall that $\eta_i = 0$ for all $i = 1, \ldots, 6$), (4.7) and (4.8), we obtain the following expressions for the Lie brackets at $(R, s) = (\mathrm{Id}, 0)$:

$$[\mathbf{Z}_c^1,\, \mathbf{Z}_c^2] = \frac{I_1 I_2(-\mu_1 - \mu_2 + \mu_3) + \mu_1\mu_2(-I_1 - I_2 + I_3)}{(\mu_1 + I_1)(\mu_2 + I_2)(\mu_3 + I_3)} \begin{pmatrix} \hat{\mathbf{e}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$[\mathbf{Z}_c^1,\, \mathbf{Z}_c^3] = \frac{I_1 I_3(\mu_1 - \mu_2 + \mu_3) + \mu_1\mu_3(I_1 - I_2 + I_3)}{(\mu_1 + I_1)(\mu_2 + I_2)(\mu_3 + I_3)} \begin{pmatrix} \hat{\mathbf{e}}_2 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$[\mathbf{Z}_c^2,\, \mathbf{Z}_c^3] = \frac{I_2 I_3(\mu_1 - \mu_2 - \mu_3) + \mu_2\mu_3(I_1 - I_2 - I_3)}{(\mu_1 + I_1)(\mu_2 + I_2)(\mu_3 + I_3)} \begin{pmatrix} \hat{\mathbf{e}}_1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$[\mathbf{Z}_c^2,\, \mathbf{Z}_c^4] = \frac{I_2 m(\mu_4 - \mu_6)}{(\mu_2 + I_2)(\mu_4 + m)(\mu_6 + m)} \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_3 \\ \mathbf{0} \end{pmatrix},$$

$$[\mathbf{Z}_c^3,\, \mathbf{Z}_c^4] = \frac{I_3 m(\mu_5 - \mu_4)}{(\mu_3 + I_3)(\mu_4 + m)(\mu_5 + m)} \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_2 \\ \mathbf{0} \end{pmatrix},$$

$$[\mathbf{Z}_c^3,\, \mathbf{Z}_c^5] = \frac{I_3 m(\mu_5 - \mu_4)}{(\mu_3 + I_3)(\mu_4 + m)(\mu_5 + m)} \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_1 \\ \mathbf{0} \end{pmatrix}.$$

Since, on the other hand, when $(R, s) = (\mathrm{Id}, 0)$ we also have

$$\mathbf{Z}_c^i = \begin{pmatrix} -\mu_i\hat{\mathbf{e}}_i/(\mu_i + I_i) \\ \mathbf{0} \\ \mathbf{f}_i \end{pmatrix} \quad \text{if } i = 1, 2, 3, \quad \text{and} \quad \mathbf{Z}_c^i = \begin{pmatrix} \mathbf{0} \\ -\mu_i\mathbf{e}_i/(\mu_i + m) \\ \mathbf{f}_i \end{pmatrix} \quad \text{if } i = 4, 5, 6,$$

we deduce that a sufficient condition ensuring that $\dim \mathrm{Lie}_{(\mathrm{Id},\mathbf{0},0)}\mathcal{Z}(c) = 12$ is that

$$\begin{aligned}\big[I_1I_2(-\mu_1-\mu_2+\mu_3)+\mu_1\mu_2(-I_1-I_2+I_3)\big]\big[I_1I_3(\mu_1-\mu_2+\mu_3)+\mu_1\mu_3(I_1-I_2+I_3)\big]\\ \big[I_2I_3(\mu_1-\mu_2-\mu_3)+\mu_2\mu_3(I_1-I_2-I_3)\big]\big[I_2m(\mu_4-\mu_6)I_3^2m^2(\mu_5-\mu_4)^2\big]\neq 0.\end{aligned} \quad (4.9)$$

According to [10, pages 152-155], if we specialize now $\mathcal{B}^\diamond$ to be an ellipsoid, the length of the axes can be chosen in such a way that (i) it has no axis of symmetry (and hence $\mu_i > 0$ for $i = 1,\dots,6$), (ii) $\mu_4 \neq \mu_5$ and (iii) $\mu_4 \neq \mu_6$. Since $I_i > 0$ ($i = 1,2,3$) and obviously $m > 0$, the condition (4.9) reduces to:

$$\begin{aligned}\big[I_1I_2(-\mu_1-\mu_2+\mu_3)+\mu_1\mu_2(-I_1-I_2+I_3)\big]\big[I_1I_3(\mu_1-\mu_2+\mu_3)+\mu_1\mu_3(I_1-I_2+I_3)\big]\\ \big[I_2I_3(\mu_1-\mu_2-\mu_3)+\mu_2\mu_3(I_1-I_2-I_3)\big]\neq 0.\end{aligned} \quad (4.10)$$

As already mentioned, it is always possible to achieve any triplet of positive numbers (I_1, I_2, I_3) with a suitable choice of density, so whatever the values of μ_i ($i = 1,2,3$) are, it is always possible to find $\varrho \in C^0(\bar{B})^+$ such that (4.10) holds. It entails, according to the forth point of Proposition 4.4:

PROPOSITION 4.5. *For any integer $n \geq 5$, the set of all the controllable SC is an open dense subset in $\mathcal{C}(n)$.*

Notice that $n = 5$ in this Proposition (instead of $n = 6$), because we did not use the vector field $\mathbf{Z}_c^6$ in the computation of the Lie brackets.

5. Proofs of the Main Results.

Proof of Proposition 1.1. Let a control function ϑ be given in $AC([0,T], D_0^1(\mathbf{R}^3))$ and denote $\Theta := \mathrm{Id} + \vartheta$. With the notation of Lemma 3.2, at any time t the entries of the matrix $\mathbb{M}_f^r(t)$ read $\Phi(\mathbf{q})$ with $\mathbf{q} := (\vartheta, \mathcal{W})$ and $\mathcal{W} := (\mathbf{W}^1, \mathbf{W}^2)$, $\mathbf{W}^i \in \{\mathbf{e}_i \times \Theta_t, \mathbf{e}_i, i = 1,2,3\}$. We deduce that $t \in [0,T] \mapsto \mathbb{M}_f^r(t) \in \mathrm{M}(3)$ is in $AC([0,T], \mathrm{M}(3))$. To get the expression of the elements of the vector $\mathbf{N}(t)$ we only have to modify $\mathbf{W}^2$ which has to be equal to $\partial_t\vartheta_t$. It entails that $t \in [0,T] \mapsto \mathbf{N}(t) \in \mathbf{R}^6$ is in $L^1([0,T])^6$. It is quite easy to verify that the inertia tensor $\mathbb{I}(t)$ is in $AC([0,T], \mathrm{M}(3))$. Existence and uniqueness of solutions is now straightforward because $t \in [0,T] \mapsto \mathbb{M}^r(t)^{-1}\mathbf{N}(t) \in \mathbf{R}^6$ is in $AC([0,T], \mathbf{R}^6)$.

Let us address the stability result. With the same notation as in the statement of Proposition 1.1, denote $(\mathbf{\Omega}^j, \mathbf{v}^j)^t$ the left hand side of identity (1.6a) with control ϑ^j and $(\bar{\mathbf{\Omega}}, \bar{\mathbf{v}})^t$ when the control is $\bar{\vartheta}$. As $j \to +\infty$, it is clear that $(\mathbf{\Omega}^j, \mathbf{v}^j)^t \to (\bar{\mathbf{\Omega}}, \bar{\mathbf{v}})^t$ in $L^1([0,T], \mathbf{R}^6)$. Then, integrating (1.6b) between 0 and t for any $0 \leq t \leq T$, we get the estimate $\|\bar{R}(t) - R^j(t)\|_{\mathrm{M}(3)} \leq \int_0^t \|\bar{R}(s) - R^j(s)\|_{\mathrm{M}(3)}\|\bar{\mathbf{\Omega}}(s)\|_{\mathbf{R}^3} + \|\mathbf{\Omega}^j(s) - \bar{\mathbf{\Omega}}(s)\|_{\mathbf{R}^3}\mathrm{d}s$. Applying Grönwall inequality, we conclude that $R^j \to \bar{R}$ in $C([0,T], \mathrm{M}(3))$ as $j \to +\infty$ and we use again the ODE to prove that $\dot{R}^j \to \dot{\bar{R}}$ in $L^1([0,T])$. Next, it is then easy to obtain the convergence of $\mathbf{r}^j$ to $\bar{\mathbf{r}}$ and to conclude the proof.

Proof of Theorems 1.2 and 1.3. We shall focus on the proof of Theorem 1.2 because it will contain the proof of Theorem 1.3. For any integer n, we shall use the notation $\|c\|_{\mathcal{C}(n)} := \|\varrho\|_{C^0(\bar{B})} + \|\vartheta\|_{C_0^1(\mathbf{R}^3)^3} + \sum_{i=1}^n \|\mathbf{V}_i\|_{C_0^1(\mathbf{R}^3)^3}$ for all $c \in C^0(\bar{B}) \times C_0^1(\mathbf{R}^3)^3 \times (C_0^1(\mathbf{R}^3)^3)^n$ with, as usual, $c := (\varrho, \vartheta, \mathcal{V})$ and $\mathcal{V} := (\mathbf{V}_1, \dots, \mathbf{V}_n)$.

Let $\varepsilon > 0$ and the functions $\bar{\varrho} \in C^0(\bar{B})$, $t \in [0,T] \mapsto \bar{\vartheta}_t \in D_0^1(\mathbf{R}^3)$ and $t \in [0,T] \mapsto (\bar{R}(t), \bar{\mathbf{r}}(t)) \in \mathrm{SO}(3) \times \mathbf{R}^3$ be given as in the statement of the theorem.

Set now $\bar{\varrho}^1 := \bar{\varrho}$, $\bar{\vartheta}^1 := \bar{\vartheta}_{t=0}$ and $\bar{\mathbf{V}}_1^1 := \partial_t\bar{\vartheta}_{t=0} \in C_0^1(\mathbf{R}^3)^3$. According to the self-propelled constraints (1.2), it is always possible to find four elements $\bar{\mathbf{V}}_j^1$ ($j = 2,\dots,5$) in $C_0^1(\mathbf{R}^3)^3$ such

that the element $\bar{c}^1 := (\bar{\varrho}^1, \bar{\vartheta}^1, \bar{\mathcal{V}}^1)$ be a SC which belongs to $\mathcal{C}(5)$, with $\bar{\mathcal{V}}^1 := (\bar{\mathbf{V}}^1_1, \ldots, \bar{\mathbf{V}}^1_5)$. Then, Proposition 4.5 guarantees that for any $\delta > 0$ it is possible to find a SC *controllable* in $\mathcal{C}(5)$, denoted by $c^1 := (\varrho^1, \vartheta^1, \mathcal{V}^1)$ where $\mathcal{V}^1 := (\mathbf{V}^1_1, \ldots, \mathbf{V}^1_5)$, such that $\|c^1 - \bar{c}^1\|_{\mathcal{C}(5)} < \delta/2$. Since $t \mapsto \partial_t \bar{\vartheta}_t$ is continuous on the compact set $[0, T]$, it is uniformly continuous. For any $\nu > 0$, there exists $\delta_\nu > 0$ such that $\|\bar{\partial}_t \vartheta_t - \bar{\partial}_t \vartheta_{t'}\|_{C^1_0(\mathbf{R}^3)^3} < \nu$ providing that $|t - t'| \leq \delta_\nu$. We then divide the time interval $[0, T]$ into $0 = t_1 < t_2 < \ldots < t_k = T$ such that $|t_{j+1} - t_j| < \delta_\nu$ for $j = 1, \ldots, k-1$. For any $t_1 \leq t \leq t_2$, we have the estimate:

$$\begin{aligned}\|\bar{\vartheta}_t - (\vartheta^1 + (t - t_1)\mathbf{V}^1_1)\|_{C^1_0(\mathbf{R}^3)^3} \leq \|\bar{\vartheta}_t - (\bar{\vartheta}^1 + (t - t_1)\bar{\mathbf{V}}^1_1)\|_{C^1_0(\mathbf{R}^3)^3} \\ + \|\bar{\vartheta}^1 - \vartheta^1\|_{C^1_0(\mathbf{R}^3)^3} + (t - t_1)\|\bar{\mathbf{V}}^1_1 - \mathbf{V}^1_1\|_{C^1_0(\mathbf{R}^3)^3}.\end{aligned}$$

On the one hande, we have, for all $t \in [t_1, t_2]$, $\|\bar{\vartheta}_t - (\bar{\vartheta}^1 + (t - t_1)\bar{\mathbf{V}}^1_1)\|_{C^1_0(\mathbf{R}^3)^3} < \nu|t - t_1|$. On the other hand, still for $t_1 \leq t \leq t_2$ and if we assume that $\delta_\nu < 1$, we get: $\|\bar{\vartheta}^1 - \vartheta^1\|_{C^1_0(\mathbf{R}^3)^3} + (t - t_1)\|\bar{\mathbf{V}}^1_1 - \mathbf{V}^1_1\|_{C^1_0(\mathbf{R}^3)^3} \leq \delta/2$. We introduce now $\bar{\varrho}^2 := \bar{\varrho}^1$ and $\bar{\vartheta}^2 := \bar{\vartheta}_{t=t_2}$. It is always possible to supplement $\bar{\mathbf{V}}^2_1 := \partial_t \bar{\vartheta}_{t_2}$ with vector fields $\bar{\mathbf{V}}^2_j$ ($j = 2, \ldots, 5$) in such a way that $\bar{c}^2 := (\bar{\varrho}^2, \bar{\vartheta}^2, \bar{\mathcal{V}}^2)$ be in $\mathcal{C}(5)$ with the obvious notation $\bar{\mathcal{V}}^2 := (\bar{\mathbf{V}}^2_1, \ldots, \bar{\mathbf{V}}^2_5)$.

We define $\varrho^2 := \varrho^1$ and $\vartheta^2 := \vartheta^1 + (t_2 - t_1)\mathbf{V}^1_1$. For any $t_1 \leq t \leq t_2$, Proposition 4.4 guarantees that the SC $c^1_t := (\varrho^1, \vartheta^1 + (t - t_1)\mathbf{V}^1_1, \mathcal{V}^1)$ is controllable. In particular, for $t = t_2$, there exists an integer k and a family of 11 vector fields in $\mathcal{E}_k$ (the set of all the Lie brackets of order lower or equal to k) such that the determinant of the family is nonzero. But this determinant can be thought of as an analytic function in $\mathcal{V}^1$. The set $\pi^{-1}(\{(\varrho^1, \vartheta^1)\})$ being an analytic connected submanifold of $(C^1_0(\mathbf{R}^3)^3)^5$ (see Corollary 2.5), the determinant is nonzero everywhere on this set but maybe in a closed subset of empty interior (for the induced topology). Therefore, it is possible to find $\mathcal{V}^2 \in (C^1_0(\mathbf{R}^3)^3)^5$ such that $\|\bar{c}^2 - c^2\|_{\mathcal{C}(5)} < (\delta/2 + \nu(t_2 - t_1)) + \delta/4$, and $c^2 := (\varrho^2, \vartheta^2, \mathcal{V}^2)$ is controllable.

By induction, we can build $\bar{c}^j$ and c^j ($j = 1, 2, \ldots, k$) such that $\|\bar{c}^j - c^j\| \leq \delta/2 + \sum_{i=2}^k \delta/2^i + \nu(t_i - t_{i-1}) < \delta + \nu T$. We choose δ and ν in such a way that $\delta + \nu T < \varepsilon/2$ and we define $t : [0, T] \mapsto \tilde{\vartheta}_t \in D^1_0(\mathbf{R}^3)$ and $t \in [0, T] \mapsto \tilde{c}_t \in \mathcal{C}(5)$ as continuous, piecewise affine functions by $\tilde{\vartheta}_t := \vartheta^j + (t - t_j)\mathbf{V}^j_1$ and $\tilde{c}_t := (\tilde{\varrho}, \tilde{\vartheta}_t, \tilde{\mathcal{V}}_t)$ with $\tilde{\varrho} := \varrho^j = \varrho^1$, $\tilde{\mathcal{V}}_t := \mathcal{V}^j$ if $t \in [t_j, t_{j+1}]$ ($j = 1, \ldots, k-1$). Notice that for any $t \in [0, T]$, (i) $\|\bar{\vartheta}_t - \tilde{\vartheta}_t\|_{C^1_0(\mathbf{R}^3)^3} < \varepsilon/2$ and (ii) $\tilde{c}_t$ is controllable.

Definition 4.2 and Proposition 4.3 ensure that, on every interval $[t_j, t_{j+1}]$ ($j = 1, \ldots, k-1$), there exist five C^1 functions $\lambda^j_i : [t_j, t_{j+1}] \mapsto \mathbf{R}$ ($i = 1, \ldots, 5$) such that the solution $(R_j, \mathbf{r}_j, s^j) : [t_j, t_{j+1}] \to \mathrm{SO}(3) \times \mathbf{R}^3 \times \mathbf{R}^5$ to the ODE (4.2) with vector fields $\mathbf{Z}^i_{c^j}(R_j, s)$ and Cauchy data $R_j(t_j) = \bar{R}(t_j)$, $\mathbf{r}_j(t_j) = \bar{\mathbf{r}}(t_j)$ and $s^j(t_j) = 0$ satisfy:

1. $\sup_{t \in [t_j, t_{j+1}]} \Big(\|\bar{R}(t) - R_j(t)\|_{\mathrm{M}(3)} + \|\bar{\mathbf{r}}(t) - \mathbf{r}_j(t)\|_{\mathbf{R}^3} + \|\tilde{\vartheta}_t - \vartheta^j_t\|_{C^1_0(\mathbf{R}^3)^3} \Big) < \varepsilon/2$ with $\vartheta^j_t := \vartheta^j + \sum_{i=1}^5 s^j_i(t)\mathbf{V}^j_i$;
2. $R_j(t_{j+1}) = \bar{R}(t_{j+1})$, $\mathbf{r}_j(t_{j+1}) = \bar{\mathbf{r}}(t_{j+1})$ and $s^j(t_{j+1}) = (t_{j+1} - t_j, 0, 0, 0, 0)^t$;

With these settings, the functions $t \in [0, T] \mapsto \breve{\vartheta}_t \in D^1_0(\mathbf{R}^3)$, $\breve{R} : [0, T] \to \mathrm{SO}(3)$ and $\breve{\mathbf{r}} : [0, T] \to \mathbf{R}^3$ defined by $\breve{\vartheta}_t := \vartheta^j_t$, $\breve{R}(t) := R_j(t)$ and $\breve{\mathbf{r}}(t) := \mathbf{r}_j(t)$ if $t \in [t_j, t_{j+1}]$ ($j = 1, \ldots, k-1$) are continuous, piecewise C^1. We extend also every functions s^j on $[0, T]$ by setting $s^j(t) := 0$ if $t \in [0, t_j[$ and $s^j(t) := (t_{j+1} - t_j, 0, 0, 0, 0)^t$ if $t \in]t_{j+1}, T]$. They are continuous, piecewise C^1 as well.

It remains to smooth the function $\breve{\vartheta}_t$ (and hence also $\breve{R}$ and $\breve{\mathbf{r}}$). We can extend the functions λ^j_i on the whole interval $[0, T]$ by merely setting $\lambda^j_i(t) = 0$ if $t \notin [t_j, t_{j+1}]$. Then, denoting $(\mathbf{F}^j_i)_{\substack{1 \leq i \leq 5 \\ 1 \leq j \leq k}}$ the canonical basis of $(\mathbf{R}^5)^k$ and $\breve{S} := \sum_{j=1}^k \sum_{i=1}^5 s^j_i \mathbf{F}^j_i \in (\mathbf{R}^5)^k$, we get that $(\breve{R}, \breve{\mathbf{r}}, \breve{S})$ is

a Carathéodory's solution to the following equation on $[0,T]$:

$$\frac{d}{dt}\begin{pmatrix}\breve{R}\\ \breve{\mathbf{r}}\\ \breve{S}\end{pmatrix}=\sum_{j=1}^{k}\sum_{i=1}^{5}\lambda_i^j(t)\mathbf{T}_i^j(\breve{R},\breve{S}),\tag{5.1}$$

where $\mathbf{T}_i^j(\breve{R},\breve{S}):=(\breve{R}\hat{\mathbf{X}}_i^1(c^j,s^j),\breve{R}\mathbf{X}_i^2(c^j,s^j),\mathbf{F}_i^j)\in T_R\mathrm{SO}(3)\times\mathbf{R}^3\times(\mathbf{R}^5)^k$. Let $\check{\lambda}_i^j$ denote analytic approximations of the functions λ_i^j in $L^1([0,T])$ and denote $(\breve{R},\breve{\mathbf{r}},\breve{S})$ the corresponding analytic solution to system (5.1) with $\breve{S}:=(\breve{s}_1^1,\ldots,\breve{s}_5^1,\breve{s}_1^2,\ldots,\breve{s}_5^2,\ldots,\breve{s}_1^k,\ldots,\breve{s}_5^k)$ and $\breve{\vartheta}_t:=\vartheta_0^1+\sum_{j=1}^k\sum_{i=1}^5\breve{s}_i^j(t)\mathbf{V}_i^j$ which is analytic from $[0,T]$ to $C_0^1(\mathbf{R}^3)^3$. According to Lemma D.2, $\breve{\vartheta}_t-\vartheta_t$ can be made arbitrarily small in $AC([0,T],C_0^1(\mathbf{R}^3)^3)$, providing that the functions $\check{\lambda}_i^j$ are close enough to λ_i^j in $L^1([0,T])$. Notice that $\breve{\vartheta}$ does probably not satisfy the self-propelled constraints (1.2) (especially at the times t_j, $j=2,\ldots,k-1$). It remains to invoke Proposition B.1 and the continuity of the input-output function (second point of Proposition 1.1) to conclude that there exists $t\in[0,T]\mapsto\vartheta_t\in D_0^1(\mathbf{R}^3)$ analytic, satisfying (1.2) and such that

$$\sup_{t\in[0,T]}\Big(\|R(t)-\breve{R}(t)\|_{\mathrm{M}(3)}+\|\mathbf{r}(t)-\breve{\mathbf{r}}(t)\|_{\mathbf{R}^3}+\|\vartheta_t-\breve{\vartheta}_t\|_{C_0^1(\mathbf{R}^3)^3}\Big)<\varepsilon/2,$$

where $(R,\mathbf{r}):[0,T]\mapsto\mathrm{SO}(3)\times\mathbf{R}^3$ is the solution to System (1.6) with initial data $(R(0),\mathbf{r}(0))=(\bar{R}(0),\bar{\mathbf{r}}(0))$ and control ϑ. The proof is then complete.

6. Conclusion. In this paper, we have proved that, for a 3D shape changing body, the ability of swimming (i.e. not only moving but tracking any given trajectory) in a vortex free environment is generic. The genericity refers to the shape of the body, its density and the basic movements (at most five) required for swimming. This result is part of a series of articles [13, 14, 15, 4] studying locomotion in a potential flow and the next step of this study will be to investigate whether this controllability result can be extended to a flow with vortices.

Appendix A. Function spaces.

Classical function spaces.

- For any compact set $K\subset\mathbf{R}^n$ (n a positive integer), the space $C^0(K)$ consists of the continuous functions in K endowed with the norm $\|u\|_{C^0(K)}:=\sup_{x\in K}|u(x)|$. The open subset of the positive functions of $C^0(K)$ is denoted $C^0(K)^+$.
- For any open set $\Omega\subset\mathbf{R}^3$ (included $\Omega=\mathbf{R}^3$), $\mathcal{D}(\Omega)$ is the space of the smooth (C^∞) functions, compactly supported in Ω.
- For any open set $\Omega\subset\mathbf{R}^3$ (included $\Omega=\mathbf{R}^3$), the set $C_0^1(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ for the norm $\|u\|_{C_0^1(\Omega)}:=\sup_{x\in\Omega}|u(x)|+\|\nabla u(x)\|_{\mathbf{R}^3}$. When $\Omega=\mathbf{R}^3$, we get $C_0^1(\mathbf{R}^3):=\{u\in C^1(\mathbf{R}^3)\,:\,|u(x)|\to 0$ and $\|\nabla u(x)\|_{\mathbf{R}^3}\to 0$ as $\|x\|_{\mathbf{R}^3}\to+\infty\}$.
- The space $C_0^1(\mathbf{R}^3)^3$ is the Banach space of all of the vector fields in $\mathbf{R}^3$ whose every component belongs to $C_0^1(\mathbf{R}^3)$.
- Let E be an open subset or an embedded submanifold of a Banach space and $T>0$, then $AC([0,T],E)$ consists in the absolutely continuous functions from $[0,T]$ into E. It is endowed with the norm $\|u\|_{AC([0,T],E)}:=\sup_{t\in[0,T]}\|u_t\|_E+\int_0^T\|\partial_t u_t\|_E\mathrm{d}t$.
- $C_0^m(\Omega,\mathrm{M}(k))$ (m an integer) is the Banach space of the functions of class C^m in $\mathbf{R}^3$ valued in $\mathrm{M}(k)$ ($\mathrm{M}(k)$ stands for the Banach space of the $k\times k$ matrices, k a positive integer) and compactly supported in Ω.

- $E_0^m(\Omega, \mathrm{M}(k))$ stands for the connected component containing the zero function of the open subset $\{M \in C_0^m(\Omega, \mathrm{M}(k)) : \det(\mathrm{Id} + M(x)) \neq 0 \ \forall\, x \in \mathbf{R}^3\}$.

LEMMA A.1. *The set* $\tilde{D}_0^1(\mathbf{R}^3) := \{\vartheta \in C_0^1(\mathbf{R}^3)^3 \text{ s.t. } \mathrm{Id} + \vartheta \text{ is a } C^1 \text{ diffeomorphism of } \mathbf{R}^3\}$ *is open in* $C_0^1(\mathbf{R}^3)^3$.

Proof. The mapping $\vartheta \in C_0^1(\mathbf{R}^3)^3 \mapsto \delta_\vartheta := \inf_{\substack{\mathbf{e} \in S^2 \\ x \in \mathbf{R}^3}} \langle \mathrm{Id} + \nabla\vartheta(x), \mathbf{e}\rangle \cdot \mathbf{e} \in \mathbf{R}$ (S^2 stands for the unit 2 dimensional sphere) is well defined and continuous. For any $\vartheta_0 \in \tilde{D}_0^1(\mathbf{R}^3)$, we have $\delta_{\vartheta_0} > 0$ and for all $x, y \in \mathbf{R}^3$ and $\mathbf{e} := (y-x)/|y-x|$ the following estimate holds: $(y + \vartheta(y) - x - \vartheta(x)) \cdot \mathbf{e} = |y-x| \int_0^1 \langle \mathrm{Id} + \nabla\vartheta(x + t\mathbf{e}), \mathbf{e}\rangle \cdot \mathbf{e}\, \mathrm{d}t > |y-x|\delta_\vartheta$. We deduce that $\mathrm{Id} + \vartheta$ is one-to-one if ϑ is close enough to ϑ_0. Further, still for ϑ close enough to ϑ_0, $\mathrm{Id} + \vartheta$ is a local diffeomorphism (according to the local inversion Theorem) and hence it is onto. ∎

DEFINITION A.2. *We denote* $D_0^1(\mathbf{R}^3)$ *the connected component of* $\tilde{D}_0^1(\mathbf{R}^3)$ *that contains the identically zero function.*

If $\vartheta \in C_0^1(\mathbf{R}^3)^3$ is such that $\|\vartheta\|_{C_0^1(\mathbf{R}^3)^3} < 1$, the local inversion Theorem and a fixed point argument ensure that $\mathrm{Id} + \vartheta$ is a C^1 diffeomorphism so we deduce that $D_0^1(\mathbf{R}^3)$ contains the unit ball of $C_0^1(\mathbf{R}^3)^3$.

Sobolev spaces. For any open exterior domain $\mathcal{F}$, the weighted Sobolev space $W^1(\mathcal{F})$ is defined by $W^1(\mathcal{F}) := \{u \in \mathcal{D}'(\mathcal{F}) \,:\, u/\sqrt{1+|x|^2} \in L^2(\mathcal{F}),\, \partial_{x_i} u \in L^2(\mathcal{F}),\, i = 1,2,3\}$ (see [3] for details).

Appendix B. Making shape changes allowable.

PROPOSITION B.1. *Let* $t \in [0,T] \mapsto \vartheta_t^j \in D_0^1(\mathbf{R}^3)$ *for* $j = 1, \ldots, +\infty$, *be a sequence of absolutely continuous functions (respectively of class* C^p *for* $p = 1, \ldots, +\infty$ *or analytic) which converges to* $t \in [0,T] \mapsto \vartheta_t^\dagger \in D_0^1(\mathbf{R}^3)$ *in* $AC([0,T], D_0^1(\mathbf{R}^3))$. *Assume that for some function* $\varrho \in C^0(\bar{B})^+$, *the pair* $(\varrho, \vartheta^\dagger)$ *satisfies* (1.2). *Then, there exists a sequence* $t \in [0,T] \mapsto \bar{\vartheta}_t^j \in D_0^1(\mathbf{R}^3)$ $(j = 1, \ldots, +\infty)$ *in* $AC([0,T], D_0^1(\mathbf{R}^3))$ *(respectively of class* C^p *for* $p = 1, \ldots, +\infty$ *or analytic) such that (i) for any* $j = 1, \ldots, +\infty$, *the pair* $(\varrho, \bar{\vartheta}^j)$ *satisfies* (1.2) *and (ii)* $\bar{\vartheta}^j \to \vartheta^\dagger$ *in* $AC([0,T], D_0^1(\mathbf{R}^3))$.

Proof. Denote $m := \int_B \varrho\, \mathrm{d}x$ and, for every $j = 1, \ldots, +\infty$, $\Theta_t^j = \mathrm{Id} + \vartheta_t^j$ and $\mathbf{r}^j(t) := (1/m)\int_B \varrho\, \Theta_t^j\, \mathrm{d}x$. Then, for any continuous function $t \in [0,T] \mapsto R(t) \in \mathrm{SO}(3)$, we have $\int_B \varrho\, R(t)\tilde{\Theta}_t^j\, \mathrm{d}x = \mathbf{0}$ where $\tilde{\Theta}_t^j := \Theta^j - \mathbf{r}^j(t)$. Let us now determine an absolutely continuous function $t \in [0,T] \mapsto R^j(t) \in \mathrm{SO}(3)$ such that we have also $\int_B \varrho[\partial_t(R^j(t)\tilde{\Theta}_t^j) \times R^j(t)\tilde{\Theta}_t^j]\mathrm{d}x = \mathbf{0}$ for all $t \in]0,T[$. Introducing $\mathbb{I}(\tilde{\Theta}_t^j) := \int_B \varrho[|\tilde{\Theta}_t^j|^2\mathrm{Id} - \tilde{\Theta}_t^j \otimes \tilde{\Theta}_t^j]\mathrm{d}x$ (an inertia tensor with $\varrho > 0$, so always invertible) and $\mathbf{G}(\tilde{\Theta}_t^j, \partial_t\tilde{\Theta}_t^j) := \mathbb{I}(\tilde{\Theta}_t^j)^{-1}\int_B \varrho\, \partial_t\tilde{\Theta}_t^j \times \tilde{\Theta}_t^j \mathrm{d}x$, the identity above can be turned into $\dot{R}^j(t) = R^j(t)\hat{\mathbf{G}}(\tilde{\Theta}_t^j, \partial_t\tilde{\Theta}_t^j)$ $(0 < t < T)$. Since $\mathrm{SO}(3)$ is compact, this ODE supplemented with the Cauchy data $R(0) = \mathrm{Id}$ admits a unique solution defined for all $t \in [0,T]$. The solution has the same regularity as $t \in [0,T] \mapsto \vartheta_t^j \in D_0^1(\mathbf{R}^3)$. Besides, basic estimates allow to prove that $(R^j, \mathbf{r}^j) \to (\mathrm{Id}, \mathbf{0})$ in $AC([0,T], \mathrm{SO}(3) \times \mathbf{R}^3)$ and next that $\tilde{\vartheta}^j \to \vartheta^\dagger$ in $AC([0,T], C_0^1(\mathbf{R}^3)^3)$. Notice that, at this point, we probably have $\tilde{\vartheta}^j \notin D_0^1(\mathbf{R}^3)$. Let now Ω be a large ball containing $\cup_{\substack{t \in [0,T] \\ j \in \mathbf{N}}} \tilde{\Theta}_t^j(\bar{B})$ and Ω' be an even larger ball containing Ω. Consider χ a cut-off function such that $0 \leq \chi \leq 1$, $\chi = 1$ in Ω and $\chi = 0$ in $\mathbf{R}^3 \setminus \bar{\Omega}'$. Define $\bar{\Theta}^j$ as the flow associated with the Cauchy problem $\dot{X}(t,x) = \chi(x)\partial_t\tilde{\vartheta}_t^j + (1 - \chi(x))\partial_t\vartheta^\dagger$, $X(0,x) = \Theta_{t=0}^\dagger(x)$, $(x \in \mathbf{R}^3)$ and $\bar{\vartheta}^j := \bar{\Theta}^j - \mathrm{Id}$. Since $\bar{\vartheta}_{t=0}^j = \vartheta_{t=0}^\dagger$, we deduce that $\bar{\vartheta}_t^j \in D_0^1(\mathbf{R}^3)$ for all $t \in [0,T]$ and the sequence $t \in [0,T] \mapsto \bar{\vartheta}_t^j \in D_0^1(\mathbf{R}^3)$ complies with the requirements of the Proposition. ∎

Appendix C. Making vector fields allowable. Let a triplet $(\varrho, \vartheta, \mathcal{V}) \in C^0(\bar{B})^+ \times D^1_0(\mathbf{R}^3) \times (C^m_0(\mathbf{R}^3)^3)^n$ be given such that $\int_B \varrho\, \Theta \,\mathrm{d}x = \mathbf{0}$ where $\Theta = \mathrm{Id} + \vartheta$. Recall that $\Sigma = \partial B$.

PROPOSITION C.1. *It is always possible to define new vector fields $\mathbf{V}^*_i$ in such a way that (i) $\mathbf{V}^*_i|_\Sigma = \mathbf{V}_i|_\Sigma$, (ii) $(\varrho, \vartheta, \mathcal{V}^*) \in \mathcal{C}(n)$ with $\mathcal{V}^* := (\mathbf{V}^*_1, \ldots, \mathbf{V}^*_n)$ and (iii) $\int_B \varrho\, \Theta \cdot \mathbf{V}^*_i \,\mathrm{d}x = 0$.*

Proof. Arguing like in the proof of Theorem 2.3, we can easily show that the mapping $\mathbf{W} \in C^1_0(B)^3 \mapsto (\int_B \varrho\, \mathbf{W} \,\mathrm{d}x, \int_B \varrho\, \Theta \times \mathbf{W} \,\mathrm{d}x, \int_B \varrho\, \Theta \cdot \mathbf{W} \,\mathrm{d}x) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$ is onto with infinite dimensional kernel. Hence, it is always possible to find a vector field $\mathbf{W}_1 \in C^1_0(B)^3$ satisfying $\int_B \varrho\, \mathbf{W}_1 \,\mathrm{d}x = -\int_B \varrho\, \mathbf{V}_1 \,\mathrm{d}x$, $\int_B \varrho\, \Theta \times \mathbf{W}_1 \,\mathrm{d}x = -\int_B \varrho\, \Theta \times \mathbf{V}_1 \,\mathrm{d}x$, $\int_B \varrho\, \Theta \cdot \mathbf{W}^1 \,\mathrm{d}x = -\int_B \varrho\, \Theta \cdot \mathbf{V}^1 \,\mathrm{d}x$ and such that $\{\Theta|_B \cdot \mathbf{e}_k, (\mathbf{V}_1 + \mathbf{W}_1)|_B \cdot \mathbf{e}_k, k = 1, 2, 3\}$ is a free family in $C^1_0(\mathbf{R}^3)^3$. We denote $\mathbf{V}^*_1 := \mathbf{V}_1 + \mathbf{W}_1$. We can continue with the same idea: The mapping $\mathbf{W} \in C^1_0(\mathbf{R}^3)^3 \mapsto (\int_B \varrho\, \mathbf{W} \,\mathrm{d}x, \int_B \varrho\, \Theta \times \mathbf{W} \,\mathrm{d}x, \int_B \varrho\, \mathbf{V}^*_1 \times \mathbf{W} \,\mathrm{d}x) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$ is onto, also with infinite dimensional kernel. Again, it is possible to find $\mathbf{W}_2 \in C^1_0(B)^3$ satisfying $\int_B \varrho\, \mathbf{W}_2 \,\mathrm{d}x = -\int_B \varrho\, \mathbf{V}_2 \,\mathrm{d}x$, $\int_B \varrho\, \Theta \times \mathbf{W}_2 \,\mathrm{d}x = -\int_B \varrho\, \Theta \times \mathbf{V}_2 \,\mathrm{d}x$, $\int_B \varrho\, \Theta \cdot \mathbf{W}_2 \,\mathrm{d}x = -\int_B \varrho\, \Theta \cdot \mathbf{V}_2 \,\mathrm{d}x$ and $\int_B \varrho\, \mathbf{V}^*_1 \times \mathbf{W}_2 \,\mathrm{d}x = -\int_B \varrho\, \mathbf{V}^*_1 \times \mathbf{V}_2 \,\mathrm{d}x$ and such that $\{\Theta|_B \cdot \mathbf{e}_k, \mathbf{V}^*_1|_B \cdot \mathbf{e}_k, (\mathbf{V}_2 + \mathbf{W}_2)|_B \cdot \mathbf{e}_k, k = 1, 2, 3\}$ is free in $C^1_0(\mathbf{R}^3)^3$. We can set $\mathbf{V}^*_2 := \mathbf{V}_2 + \mathbf{W}_2$ and iterate this process to define $\mathbf{V}^*_3, \ldots, \mathbf{V}^*_n$. ☐

Appendix D. Added mass matrix for particular shaped swimmers. Let $\mathcal{B}$ be an open, bounded, connected, C^1 subset of $\mathbf{R}^3$. Denote Σ its boundary, $\mathbf{n}$ the unit vector to Σ directed toward the interior of $\mathcal{B}$, $\mathcal{F} := \mathbf{R}^3 \setminus \bar{\mathcal{B}}$ and consider the 6×6 symmetric matrix $\mathbb{M}^f$ of which the entries are defined by $\int_{\mathcal{F}} \nabla\psi_i \cdot \nabla\psi_j \,\mathrm{d}x$ $(1 \leq i, j \leq 6)$ where the functions ψ_i $(i = 1, \ldots, 6)$ are harmonic in $\mathcal{F}$ and satisfy the Neumann boundary conditions $\partial_n \psi_i = (\mathbf{e}_i \times x) \cdot \mathbf{n}$ on Σ if $i = 1, \ldots, 6$ and $\partial_n \psi_i = \mathbf{e}_{i-3} \cdot \mathbf{n}$ if $i = 4, 5, 6$.

PROPOSITION D.1. *The matrix $\mathbb{M}^f$ is always positive and it is positive definite if and only if $\mathcal{B}$ has no axis of symmetry.*

Proof. Denote $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_6)^t$ any element in $\mathbf{R}^6$ and $\psi := \sum_{i=1}^6 \alpha_i \psi_i$. Then, we have $\boldsymbol{\alpha}^t \mathbb{M}^f \boldsymbol{\alpha} = \int_{\mathcal{F}} \|\nabla\psi\|^2_{\mathbf{R}^3} \mathrm{d}x \geq 0$ which proves that $\mathbb{M}^f$ is indeed positive. Let now $\boldsymbol{\alpha}$ be in $\mathbf{R}^6$ such that $\boldsymbol{\alpha}^t \mathbb{M}^f \boldsymbol{\alpha} = 0$. It means that $\psi = 0$ and hence $\partial_n \psi = \sum_{i=1}^3 \alpha_i (\mathbf{e}_i \times x) \cdot \mathbf{n} + \sum_{i=4}^6 \alpha_i \mathbf{e}_i \cdot \mathbf{n} = 0$. Denoting $\mathbf{u} := \sum_{i=1}^3 \alpha_i \mathbf{e}_i$ and $\mathbf{v} := \sum_{i=4}^6 \alpha_i \mathbf{e}_i \cdot \mathbf{n} = 0$, this condition reads $(\mathbf{u} \times x + \mathbf{v}) \cdot \mathbf{n} = 0$ on Σ. The case $\mathbf{u} = \mathbf{0}$ is obviously not possible, otherwise we would have $\mathbf{v} \cdot \mathbf{n}(x) = 0$ for all $x \in \Sigma$ so let us assume from now on that $\mathbf{u} \neq \mathbf{0}$. For all $x \in \Sigma$, $(\mathbf{u} \times x + \mathbf{v}) \in T_x\Sigma$ (the tangent space to Σ at x). Let us set $\mathbf{w} := (\mathbf{u} \times \mathbf{v})/|\mathbf{u}|^2$. Then $(\mathbf{u} \times x + \mathbf{v}) = \mathbf{u} \times (x - \mathbf{w}) + \lambda \mathbf{u}$ with $\lambda := (\mathbf{v} \cdot \mathbf{u})/|\mathbf{u}|^2$. In this form, we can explicitly compute the expression of the flow connecting to the ODE $\dot{X}(t, x) = \mathbf{u} \times (X(t, x) - \mathbf{w}) + \lambda \mathbf{u}$, $X(0, x) = x$. We obtain $X(t, x) = \exp(t\hat{\mathbf{u}})(x - \mathbf{w}) + \mathbf{w} + t\lambda \mathbf{u}$ for all $t \in \mathbf{R}$. Since $X(t, x)$ has to remain on Σ for all $t \in \mathbf{R}$ and Σ is bounded, we deduce that $\lambda \mathbf{u} = \mathbf{0}$. For all $x \in \Sigma$ and all $t \in \mathbf{R}$, the point $X(t, x)$ lies on Σ. It means that Σ is globally invariant under a rotation whose axis has $\mathbf{u}$ as direction vector and goes through $\mathbf{w}$. ☐

Let $\mathcal{M}$ be a smooth embedded submanifold of a Banach space E and $\mathcal{X} := (\mathbf{X}_i)_{1 \leq i \leq n}$ be set of smooth vector fields on $\mathcal{M}$.

LEMMA D.2. *Let α_i be in $L^\infty([0, T])$ $(i = 1, \ldots, n)$ and x be a Carathéodory's solution defined on the time interval $[0, T]$ to the ODE $\dot{x}(t) = \sum_{i=1}^n \alpha_i(t) \mathbf{X}_i(x)$ $(0 < t \leq T)$, with Cauchy data $x(0) = x_0 \in \mathcal{M}$. Let the functions α^k_i , $(i = 1, \ldots, n$, $k \in \mathbf{N})$ be in $L^1([0, T])$, such that $\alpha^k_i \to \alpha_i$ in $L^1([0, T])$ as $k \to +\infty$. Then, for any sequence of Carathéodory's solutions $(x_k)_{k \in \mathbf{N}}$ satisfying $\dot{x}_k(t) = \sum_{i=1}^n \alpha^k_i(t) \mathbf{X}_i(x)$ with Cauchy data $x_k(0) = x_0$, the functions x_k can be continued on the whole interval $[0, T]$ for k large enough, $x_k \to x$ uniformly on $[0, T]$ and $\dot{x}_k \to \dot{x}$ in $L^1([0, T], \mathcal{M})$.*

Proof. For any $\delta > 0$ small enough, denote by K_δ the compact $\{x \in \mathcal{M} : \|x - x(t)\|_E \leq \delta, t \in [0, T]\}$ and denote $k_\delta > 0$ the Lipschitz constant such that $\sum_{i=1}^n \|\mathbf{X}_i(x) - \mathbf{X}_i(y)\|_E < k_\delta \|x - y\|_E$

for all $x, y \in K_\delta$ and all $i = 1, \dots, n$. Let $M := \max_{\substack{x\in K_\delta \\ i=1,\dots,n}} \|\mathbf{X}_i(x)\|_E$ and $m := \sup_{\substack{t\in[0,T] \\ i=1,\dots,n}} |\alpha_i(t)|$. Any function x_k is defined at least on a small interval $[0, t_k[$ and we can choose t_k small enough such that $x_k(t) \in K_\delta$ for all $t \in [0, t_k[$. We get the estimate, for all $t \in [0, t_k[$:

$$\|x(t) - x_k(t)\|_E \leq M \sum_{i=1}^{n} \int_0^t |\alpha_i(s) - \alpha_i^k(s)|\mathrm{d}s + mk_\delta \int_0^t \|x_k(s) - x(s)\|_E \mathrm{d}s. \tag{D.1}$$

For any $\varepsilon > 0$, we can choose k large enough such that $\sum_{i=1}^n \int_0^t |\alpha_i(s) - \alpha_i^k(s)|\mathrm{d}s < \varepsilon e^{-mk_\delta T}/M$. Applying Grönwall inequality to (D.1), we obtain that $\|x(t) - x_k(t)\|_E < \varepsilon$ for all $t \in [0, t_k[$. We deduce first that if $\varepsilon < \delta$, the solution x_k can be continued on the whole interval $[0, T]$ and then that $x_k \to x$ uniformly as $k \to +\infty$. Writing now that:

$$\begin{aligned}\int_0^t \|\dot{x}_k(t) - \dot{x}(t)\|_E \mathrm{d}t &\leq \sum_{i=1}^{n} \int_0^t |\alpha_i^k(t) - \alpha_i(t)| \|\mathbf{X}_i(x_k(s))\|_E + |\alpha_i(t)| \|\mathbf{X}_i(x_k(t)) - \mathbf{X}_i(x(t))\|_E \mathrm{d}t \\ &\leq M \sum_{i=1}^{n} \int_0^t |\alpha_i^k(t) - \alpha_i(t)|\mathrm{d}t + mk_\delta \int_0^t \|x_k(t) - x(t)\|_E \mathrm{d}t,\end{aligned}$$

we get the convergence of the sequence $(\dot{x}_k)_{k\in\mathbf{N}}$ to $\dot{x}$ in $L^1([0, T], \mathcal{M})$. ◻

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